

Dynamical renormalization group resummation of finite temperature infrared divergences

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We introduce the method of dynamical renormalization group to study relaxation and damping out of equilibrium directly in real time and apply it to the study of infrared divergences in scalar QED. This method allows a consistent resummation of infrared effects associated with the exchange of quasistatic transverse photons and leads to anomalous logarithmic relaxation of the form $e^{-\alpha T \ln[t/t_0]}$ for hard momentum charged excitations. This is in contrast with the usual quasiparticle interpretation of charged collective excitations at finite temperature in the sense of exponential relaxation of a narrow width resonance for which the width is the imaginary part of the self-energy on shell. In the case of narrow resonances away from thresholds, this approach leads to the usual exponential relaxation. The hard thermal loop resummation program is incorporated consistently into the dynamical renormalization group yielding a picture of relaxation and damping phenomena in a plasma in real time that transcends the conceptual limitations of the quasiparticle picture and other types of resummation schemes. [S0556-2821(99)02516-3]

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I. INTRODUCTION

The possibility of studying experimentally the formation and evolution of the quark gluon plasma at the BNL Relativistic Heavy Ion Collider (RHIC) and CERN Large Hadron Collider (LHC) motivates a deeper understanding of collective excitations in ultrarelativistic plasmas (for reviews see [1–10]). An important concept in the assessment of whether the quark gluon plasma achieves local thermodynamic equilibrium is that of the relaxation time scale or damping rate which determines the lifetime of excitations in the plasma [9–12]. The concept and definition of the damping rate of a collective excitation is associated with a quasiparticle description of these excitations in the plasma and imply exponential relaxation. The validity of the quasiparticle concept requires that the lifetime must be large compared to the oscillation period of the particular excitation mode. In this quasiparticle picture the collective excitations are described as narrow resonances, their spectral function is of the Breit-Wigner form and the damping rate is obtained from the width of this resonance. For weakly interacting quasiparticles the narrow resonance (quasiparticle) approximation is expected to be reliable and the damping rate or lifetime is obtained from the imaginary part of the self-energy on the mass shell of the collective excitation [9–12].

Early attempts to calculate the damping rates of quasiparticles in lowest order perturbation theory obtained gauge dependent and unphysical results [13]. Braaten and Pisarski [14–18] introduced a resummation scheme [the resummation of the hard thermal loops (HTL)] that incorporates the screening corrections in a gauge invariant manner and render finite transport cross sections [19]. These hard thermal loop screening corrections are sufficient to render finite the damping rate of excitations at rest in the plasma. However, these screening corrections are not sufficient to cure the infrared

divergences in the damping rate of charged excitations at non-zero and large (hard) spatial momentum [20]. The infrared divergences arise from the emission and absorption of long-wavelength magnetic (transverse) photons or gluons which are not screened by the hard thermal loop corrections [20]. Whereas longitudinal photons (instantaneous Coulomb interaction) are screened at finite temperature with a Debye screening mass $\approx eT$, magnetic photons (transverse) are dynamically screened for non-zero frequency as a result of Landau damping [13–16,18,20]. However, quasistatic long-wavelength magnetic photons are not screened in the Abelian theory, and their emission and absorption by a fast moving charged particle results in infrared divergences in the imaginary part of the self-energy on shell.

These infrared divergences for charged particles are not specific to particular theories but are somewhat universal in the sense that the same structure of divergences is common to QED, QCD and scalar QED in lowest order in the HTL resummation [21–29]. Further studies of the spectral function questioned the validity of the quasiparticle approximation and the exponential relaxation associated with a damping rate [30,31]. Although these studies provided an understanding of the failure of the quasiparticle picture (exponential relaxation) for hard fermions, the issue of the relaxation time scales was only recently clarified by the implementation of a Bloch-Nordsieck resummation of the infrared divergent diagrams [32,33] which yields anomalous logarithmic relaxation. This resummation scheme was previously used at finite temperature to verify the cancellation of infrared divergences of soft photons [34].

Infrared divergences in the propagators of charged fields are not particular of finite temperature field theory. It is well known that electrons in QED do not have a pole associated with their mass shell but rather a cut structure. This is a consequence of the emission of soft photons which because

of their masslessness make the putative electron mass shell pole to become the beginning of a cut. This results in that, even after ultraviolet renormalization, the wave function renormalization is logarithmically infrared divergent on shell. The implementation of the Bloch-Nordsieck resummation of these infrared divergences at zero temperature leads to the correct electron propagator [35,36]. This resummation at zero temperature is equivalent to a renormalization group resummation of the leading infrared divergences in the Euclidean fermion propagator and leads to an anomalous scaling dimension (albeit gauge dependent) for the threshold behavior of fermions in QED [35].

In this article we introduce a *dynamical renormalization group* resummation program that allows one to obtain the *real time* dependence of retarded propagators, and leads unequivocally to the real time dynamics of relaxation and thermalization without any assumptions on quasiparticle structure of collective excitations. This resummation scheme is the dynamical (real time) equivalent of the renormalization group resummation of divergences in Euclidean Green's functions which is so successful in both critical phenomena and asymptotic freedom and transcends approximations of the Bloch-Nordsieck type. The main concept in this program is the resummation of secular terms in the perturbative solution of the equation of evolution of *expectation values* that determine the real time retarded propagators. This dynamical renormalization group was originally developed to improve the solutions of ordinary (and partial) differential equations [37], and has been recently implemented in quantum field theory out of equilibrium [38,39] where it reveals relaxation with anomalous (and non-perturbative) exponents [39] (for other applications in quantum mechanics of few degrees of freedom see [40]).

In the real time description of the dynamical evolution, the time variable acts as an infrared cutoff. The infrared singularities associated with the absorption and emission of massless quanta are manifest as logarithmic secular terms in the perturbative solution of the initial value problem. The dynamical renormalization group implements a non-perturbative *resummation* of these secular terms and leads to anomalous relaxation. In particular for scalar QED we find that the charged scalar field expectation value with hard momentum relaxes in absolute value as $e^{-\alpha T t \ln[t/t_0]}$ at asymptotically long times. The asymptotic relaxation is determined by the behavior of the density of states $\rho(k; \omega)$ as a function of ω near threshold ($\omega = \sqrt{k^2 + m^2}$). The larger is $\rho(k; \omega)$ there, the faster is the decay of the expectation value of the field.

The advantage of this method is that it leads to an understanding of relaxation *directly in real time* displaying clearly the contributions from different regions of the spectral density to the long time behavior. Furthermore, it offers a simple criterion to distinguish exponential relaxation and more complicated relaxational phenomena that cannot be interpreted within the quasiparticle picture.

This method implements renormalization group resummations without the need for invoking a quasiparticle picture or any other approximation.

The article is organized as follows. In Sec. II we provide

a direct link between linear response and the study of relaxation phenomena as an initial value problem out of equilibrium. In Sec. III we introduce and test the method of dynamical renormalization group within the simple setting of a field theory of two interacting scalars, one heavy and the other massless. This simpler theory presents the same type of infrared threshold singularities as scalar QED and QED. This model presents the infrared threshold divergences of a critical theory at the upper critical dimensionality. In this section we compare the Bloch-Nordsieck approximation and the renormalization group resummation of infrared divergences in the *Euclidean* propagator to the real time resummation implemented by the dynamical renormalization approach in different situations at zero temperature. This study shows in detail the equivalence of all the different approaches at $T = 0$. We then implement the dynamical renormalization group at finite temperature and find anomalous logarithmic relaxation as in finite temperature QED [32]. Section IV is devoted to a discussion of the dynamical renormalization group to elucidate this resummation program and to make contact with the usual renormalization in Euclidean space-time. In Sec. V we study in detail scalar quantum electrodynamics. This theory has been previously studied within the imaginary time, equilibrium formulation [29] and shown to have the same type of behavior as QED and QCD in leading order in the HTL resummation. We study both the exchange of bare photons and include the HTL resummation program consistently to leading order into the dynamical renormalization group. This combined resummation of HTL and infrared secular terms in real time leads at once to anomalous logarithmic relaxation as in QED in the Bloch-Nordsieck approximation [32] and in the simpler scalar case studied in Sec. III.

We summarize our studies in the conclusion wherein we advocate to use this new approach based on the dynamical renormalization group to study fermionic excitations in a plasma and raise further questions and comments. The method of dynamical renormalization group leads directly to an understanding of damping and relaxation *in real time* without invoking a quasiparticle picture or any other approximation.

Two appendixes provide technical details and a third appendix provides a very simple and pedagogical example of the dynamical renormalization group.

II. PRELIMINARIES: FROM LINEAR RESPONSE TO INITIAL VALUE PROBLEM

We are interested in studying the real time evolution of expectation values of field operators. Consider a scalar field theory with an interacting Lagrangian density $\mathcal{L}[\Phi]$ the expectation value of the scalar field Φ can be obtained from linear response to an external c -number source term J . The appropriate formulation of real time, non-equilibrium dynamics is that of Schwinger and Keldysh [41–44] in which a path integral along a contour in imaginary time is required to generate all of the non-equilibrium Green's functions.

The non-equilibrium Lagrangian density along this contour is therefore given by [41–44]

$$\mathcal{L}_{NEQ}[\Phi^+, \Phi^-, J] = \mathcal{L}[\Phi^+] + J\Phi^+ - \mathcal{L}[\Phi^-] - J\Phi^-.$$

The non-equilibrium expectation value of the scalar field in a linear response analysis is given by

$$\begin{aligned} \langle \Phi^+(\vec{x}, t) \rangle &= \langle \Phi^-(\vec{x}, t) \rangle \\ &= \phi(\vec{x}, t) \\ &= \int_{-\infty}^{\infty} d^3\vec{x}' dt' G_R(\vec{x} - \vec{x}', t - t') J(\vec{x}', t') \end{aligned}$$

with the retarded Green's function

$$\begin{aligned} G_R(\vec{x} - \vec{x}', t - t') &= [G^>(\vec{x} - \vec{x}', t - t') \\ &\quad - G^<(\vec{x} - \vec{x}', t - t')] \Theta(t - t') \\ &= i[\langle \Phi(\vec{x}, t), \Phi(\vec{x}', t') \rangle] \Theta(t - t') \end{aligned}$$

where the expectation value is in the full interacting theory but with vanishing source. Consider an external source term that is adiabatically switched on in time from $t \rightarrow -\infty$ and of the form

$$J(\vec{x}', t') = J(\vec{x}') e^{\epsilon t'} \Theta(-t'), \quad \epsilon \rightarrow 0^+. \quad (2.1)$$

The retarded nature of $G_R(\vec{x} - \vec{x}', t - t')$ results in that

$$\phi(\vec{x}, t=0) = \phi_0(\vec{x}) \quad (2.2)$$

$$\dot{\phi}(\vec{x}, t < 0) = 0 \quad (2.3)$$

where $\phi_0(\vec{x})$ is determined by $J(\vec{x})$ [or, vice versa, the initial value $\phi_0(\vec{x})$ can be used to find $J(\vec{x})$] and the vanishing of the derivative for $t < 0$ is a consequence of the retarded nature of G_R . The linear response problem with the initial conditions at $t=0$ given by Eqs. (2.2) and (2.3) can now be turned into an initial value problem for the *equation of motion* of the expectation value by using the (integro-)differential operator $\mathcal{O}_{(\vec{x}, t)}$ inverse of $G_R(\vec{x} - \vec{x}', t - t')$,

$$\begin{aligned} \mathcal{O}_{(\vec{x}, t)} \phi(\vec{x}, t) &= J(\vec{x}, t), \quad \phi(\vec{x}, t=0) = \phi_0(\vec{x}), \\ \dot{\phi}(\vec{x}, t < 0) &= 0, \end{aligned}$$

for the source term given by Eq. (2.1). Within the non-equilibrium formulation the equation of motion of the expectation value is obtained via the tadpole method and automatically leads to a retarded initial value problem by coupling an external source that satisfies Eq. (2.1).

III. SIMPLE EXAMPLE: A SCALAR THEORY

We begin by considering a simple scalar theory of a massive and a massless scalar field with Lagrangian density

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \sigma)^2 + \frac{1}{2}(\partial_\mu \pi)^2 - \frac{1}{2}m_0^2 \sigma^2 - g_0 \sigma^2 \pi + J_0 \sigma$$

where the source coupled to the sigma field has been introduced to provide an initial value problem as explained in the previous section. Introducing the following renormalizations

$$\begin{aligned} \sigma &= Z_\sigma^{1/2} \sigma_r, \quad \pi = Z_\pi^{1/2} \pi_r, \quad J_r = Z_\sigma^{1/2} J_0, \\ m_0^2 Z_\sigma &= m_r^2 Z_\sigma + \delta m^2, \quad g_0 Z_\sigma Z_\pi^{1/2} = g_r Z_g. \end{aligned}$$

We now suppress the label r with all quantities being renormalized, and write the Lagrangian density in terms of renormalized quantities and counterterms

$$\begin{aligned} \mathcal{L} &= \frac{1}{2}(\partial_\mu \sigma)^2 + \frac{1}{2}(\partial_\mu \pi)^2 - \frac{1}{2}m^2 \sigma^2 - g \sigma^2 \pi + J \sigma + \mathcal{L}_{ct} \\ \mathcal{L}_{ct} &= \frac{1}{2}(Z_\sigma - 1)[(\partial_\mu \sigma)^2 - m^2 \sigma^2] - \frac{1}{2} \delta m^2 \sigma^2 + \frac{1}{2}(Z_\pi - 1) \\ &\quad \times (\partial_\mu \pi)^2 - \frac{1}{2} m^2 \sigma^2 - g(Z_g - 1) \sigma^2 \pi - \frac{1}{2} \delta m_\pi^2 \pi^2 \end{aligned}$$

where we introduced a mass counterterm for the π field to keep it massless. The counterterms are adjusted in perturbation theory as usual.

The purpose of studying this simpler model is twofold: (i) It provides a simpler setting to implement and test the method of the dynamical renormalization group and compare with previous studies of relaxation [43,44]. (ii) The nature of the infrared divergences in this simpler theory is similar to that of gauge theories in lowest order, i.e. the exchange of a massless field in the self-energy of a massive field. These infrared divergences are very similar to those found in scalar QED [29], QED [32] and lowest order QCD [10]. After studying the resummation of the infrared divergences via the dynamical renormalization group, we apply the method to gauge theories.

Writing $\sigma(\vec{x}, t) = \psi(\vec{x}, t) + \phi(\vec{x}, t)$ with $\langle \psi(\vec{x}, t) \rangle = 0$; $\phi(\vec{x}, t) = \langle \sigma(\vec{x}, t) \rangle$ using the tadpole condition and taking spatial Fourier transforms we find the equation of motion in the amplitude expansion:

$$\begin{aligned} \ddot{\phi}_k(t) + (Z_\sigma - 1)[\ddot{\phi}_k(t) + \omega_k^2 \phi_k(t)] + [\omega_k^2 + \delta m^2] \phi_k(t) \\ + \int_{-\infty}^t \Sigma_k(t - t') \phi_k(t') dt' = J_k(t) \end{aligned}$$

with $\omega_k^2 = k^2 + m^2$. We have absorbed the contribution of a momentum and time independent tadpole (ultraviolet and infrared divergent) in a renormalization of the mass. As discussed in the previous section, the source is chosen so that $\phi_k(t=0) = \phi_k(0)$, $\dot{\phi}_k(t \leq 0) = 0$. $\Sigma(t - t')$ is the retarded self-energy. Writing $\dot{\Sigma}_k(t - t') = (d/dt') \gamma_k(t - t')$ [with the boundary condition $\gamma_k(-\infty) = 0$ corresponding to adiabatic switching on of the interaction] and imposing that $J_k(t > 0) = 0$, after an integration by parts the equation of motion for $t > 0$ becomes

$$\begin{aligned} & \ddot{\phi}_k(t) + (Z_\sigma - 1)[\ddot{\phi}_k(t) + \omega_k^2 \phi_k(t)] \\ & + [\omega_k^2 + \delta m^2 + \gamma_k(0)]\phi_k(t) \\ & - \int_0^t \gamma_k(t-t')\dot{\phi}_k(t') dt' = 0. \end{aligned} \quad (3.1)$$

To one loop order we find

$$\begin{aligned} \Sigma_k(t-t') = & -\frac{g^2}{4\pi^3} \int \frac{d^3q}{q\omega_{k+q}} \{ (1 + N_q + n_{k+q}) \\ & \times \sin[(q + \omega_{k+q})(t-t')] - (N_q - n_{k+q}) \\ & \times \sin[(q - \omega_{k+q})(t-t')] \} \end{aligned} \quad (3.2)$$

where N_q is the Bose-Einstein distribution function for the massless field π and n_{k+q} is the corresponding distribution for the massive field σ . It proves convenient to write the self-energy in the form of a dispersion relation

$$\Sigma_k(t-t') = \int_{-\infty}^{\infty} d\omega \rho(k; \omega) \sin[\omega(t-t')] \quad (3.3)$$

$$\begin{aligned} \rho(k; \omega) = & -\frac{g^2}{4\pi^3} \int \frac{d^3q}{q\omega_{k+q}} \{ (1 + N_q + n_{k+q}) \delta(\omega - q - \omega_{k+q}) \\ & - (N_q - n_{k+q}) \delta(\omega - q + \omega_{k+q}) \} \end{aligned} \quad (3.4)$$

and consequently

$$\gamma_k(t) = \int \frac{d\omega}{\omega} \rho(k; \omega) \cos(\omega t). \quad (3.5)$$

A simple calculation yields

$$\begin{aligned} \rho(k; \omega) = & -\frac{g^2}{2\pi^2} \left\{ \left[\frac{\omega^2 - \omega_k^2}{\omega^2 - k^2} \right. \right. \\ & + \frac{T}{k} \ln \frac{(1 - e^{-q^+/T})(1 - e^{-(\omega - q^-)/T})}{(1 - e^{-q^-/T})(1 - e^{-(\omega - q^+)/T})} \Big] \Theta(\omega - \omega_k) \\ & \left. + \frac{T}{k} \ln \frac{(1 - e^{-q^+}/T)}{(1 - e^{-(q^+ - \omega)/T})} \right\} [\Theta(k - \omega) + \Theta(k + \omega)], \end{aligned} \quad (3.6)$$

$$q^\pm = \frac{\omega^2 - \omega_k^2}{2(\omega \mp k)}.$$

There are some noteworthy features of the spectral density (3.6) above: (i) Whereas the zero temperature contribution vanishes (linearly) near threshold at $\omega = \omega_k$, the finite temperature contribution *does not* vanish at threshold. (ii) The finite temperature contribution below the light cone $-k < \omega < k$ has its origin in Landau damping type processes in which the σ particle scatters off a π particle in the medium.

As it will be seen and understood below the spectral density (3.6) leads to threshold infrared divergences. These will be studied in detail for the cases $T=0$, $T \neq 0$ separately in the next subsections.

The equation of motion (3.1) can be solved by Laplace transform. In terms of the Laplace transforms of $\phi_k(t)$ and $\gamma_k(t)$ given by $\tilde{\phi}_k(s)$ and $\tilde{\gamma}_k(s)$ respectively, with s the Laplace transform variable, we find

$$\tilde{\phi}_k(s) = \frac{\phi_k(0)}{s} \left\{ 1 - \frac{\omega_k^2 \mathcal{C}}{s^2 + \omega_k^2 + \Pi(s)} \right\} \quad (3.7)$$

with

$$\mathcal{C} = 1 + (Z_\sigma - 1) + [\delta m^2 + \gamma_k(0)]/\omega_k^2 \quad (3.8)$$

$$\Pi(s) = (s^2 + \omega_k^2)(Z_\sigma - 1) + \delta m^2 + \gamma_k(0) - s\tilde{\gamma}_k(s).$$

The term

$$G(k, s) = [s^2 + \omega_k^2 + \Pi(s)]^{-1}$$

is recognized as the propagator in terms of the Laplace variable s . The retarded Green's function is obtained by the analytic continuation

$$G_{ret}(k, \omega) = G(k, s = i\omega + \epsilon)|_{\epsilon=0^+}.$$

The Laplace transform of the self-energy is recognized to be

$$\tilde{\Sigma}_k(s) = \gamma_k(0) - s\tilde{\gamma}_k(s) = \int d\omega \rho(\omega) \frac{\omega}{s^2 + \omega^2}$$

and its analytic continuation is then given by

$$\begin{aligned} \tilde{\Sigma}(s = i\omega + 0^+) = & \Sigma_R(\omega) + i\Sigma_I(\omega) \\ \Sigma_R(\omega) = & \int d\omega' \rho(k, \omega') \mathcal{P} \frac{\omega'}{\omega'^2 - \omega^2} \\ \Sigma_I(\omega) = & -\frac{\pi}{2} [\rho(k, |\omega|) - \rho(k, -|\omega|)] \text{sgn}(\omega). \end{aligned} \quad (3.9)$$

Therefore $\Pi(s)$ is recognized as the twice subtracted self-energy which is rendered finite by a proper choice of counterterms. Furthermore, choosing to renormalize at $s^2 = -\omega_k^2$ with the counterterms given by

$$\begin{aligned} Z_\sigma - 1 = & \left. \frac{\partial \Sigma_R(k, \omega)}{\partial \omega^2} \right|_{\omega=\omega_k} \\ \delta m^2 = & - \int \rho(k, \omega') \mathcal{P} \frac{\omega'}{\omega'^2 - \omega_k^2} d\omega' \end{aligned}$$

we find that

$$\mathcal{C} = 1 + \int \frac{\rho(k, \omega')}{\omega'} \mathcal{P} \frac{\omega_k^2}{(\omega'^2 - \omega_k^2)^2} d\omega' \quad (3.11)$$

is *finite* even for renormalizable theories in which $\rho(k, \omega) \sim \omega^2$ at large ω , leading to quadratic and logarithmic divergences with logarithmically divergent wave function renormalizations. This *finite* wave function renormalization will be seen to emerge naturally from the dynamical renormalization group.

The real time evolution is obtained by performing the inverse Laplace transform along a path in the complex- s plane parallel to the imaginary axis to the right of all the singularities of $\tilde{\mathcal{G}}_k(s)$. We note that the putative pole at $s=0$ has vanishing residue.

A. Bloch-Nordsieck and renormalization group at $T=0$

At $T=0$ we find after renormalizing the mass

$$\tilde{\Sigma}(s) = \gamma_k(0) - s \tilde{\gamma}_k(s) = \frac{g^2}{4\pi^2} \frac{P_E^2 + m^2}{P_E^2} \ln \frac{P_E^2 + m^2}{m^2} \quad (3.12)$$

$$P_E^2 = s^2 + k^2$$

where P_E^2 is identified with the *Euclidean* four momentum squared. Since in this theory the wave function renormalization is finite, we choose $Z_\sigma = 1$ in what follows. Therefore up to this one loop order we obtain the Euclidean irreducible two point function (the inverse of the Green's function) to be given by

$$\Gamma(k, s) = [G(k, s)]^{-1} = m_R^2 \left(\frac{P_E^2}{m_R^2} + 1 \right) \times \left[1 + \frac{g^2}{4\pi^2 P_E^2} \ln \left(\frac{P_E^2}{m_R^2} + 1 \right) \right]. \quad (3.13)$$

We clearly see that $P_E^2 \rightarrow -m_R^2$ is no longer a pole but the end point of a logarithmic branch cut, corresponding to the threshold for the intermediate state of a σ particle and a soft massless π particle.

This logarithmic infrared divergence near threshold is the same phenomenon as in gauge theories wherein the self-energy of charged fields has an infrared logarithmic divergence at threshold associated with the emission of soft quanta. The Bloch-Nordsieck resummation exponentiates the logarithmic divergences near threshold leading to

$$G_{BN}(k, s) = \frac{1}{m_R^2} \left(\frac{P_E^2}{m_R^2} + 1 \right)^{\lambda-1} \Big|_{P_E^2/m_R^2 \approx -1} \quad (3.14)$$

with the dimensionless coupling

$$\lambda = \frac{g^2}{4\pi^2 m_R^2}. \quad (3.15)$$

Using this resummed expression for the propagator the inverse Laplace transform can be performed by wrapping the contour around the branch cuts along the imaginary axis from $s = \pm i\omega_k$ to $\pm i\infty$. Computing the discontinuity of $G_{BN}(k, s)$ across these cuts the real time evolution of the expectation value is given by

$$\phi_k(t) = \frac{\phi_k(0)}{m^2} \omega_k^2 \mathcal{C} m^{2(1-\lambda)} \frac{2}{\pi} \sin[\pi\lambda] \int_{\omega_k}^{\infty} \frac{d\omega}{\omega} \frac{\cos(\omega t)}{[\omega^2 - \omega_k^2]^{1-\lambda}}$$

with $\omega_k^2 = k^2 + m^2$.

Using now the result [46]

$$\begin{aligned} & \int_1^{\infty} \frac{dx \cos(xy)}{x(x^2-1)^{1-\lambda}} \\ &= \frac{\pi^2 y}{4 \sin \pi \lambda} \{ J_{1/2-\lambda}(y) [N_{-1/2-\lambda}(y) \\ & \quad - \mathcal{H}_{-1/2-\lambda}(y)] \\ & \quad - J_{-1/2-\lambda}(y) [N_{1/2-\lambda}(y) - \mathcal{H}_{1/2-\lambda}(y)] \} \end{aligned}$$

with $J_\nu(z)$ a Bessel function and $\mathcal{H}_\nu(z)$ a Struve function, we find the asymptotic long time behavior

$$\begin{aligned} \phi_k(t) &\stackrel{t \rightarrow \infty}{=} -\frac{\phi_k(0)}{m^2} \frac{\omega_k^2 \mathcal{C}}{\Gamma(1-\lambda)} \left[\frac{m^2}{\omega_k^2} \right]^{1-\lambda} \left[\frac{2}{\omega_k t} \right]^\lambda \\ &\quad \times \cos \left(\omega_k t + \frac{\pi\lambda}{2} \right) \left[1 + \mathcal{O} \left(\frac{1}{\omega_k t} \right) \right]. \end{aligned} \quad (3.16)$$

Here we used the asymptotic formula [46]

$$\mathcal{H}_\nu(z) - N_\nu(z) \stackrel{z \rightarrow \infty}{=} \frac{1}{\sqrt{\pi} \Gamma \left(\nu + \frac{1}{2} \right)} \left(\frac{z}{2} \right)^{\nu-1} \left[1 + \mathcal{O} \left(\frac{1}{z^2} \right) \right].$$

We see that the Bloch-Nordsieck resummation of the infrared divergences leads to relaxation with an *anomalous exponent*. This is similar to the case of QED. We now argue that the infrared divergence and the emergence of anomalous dimensions can be understood by establishing a parallel with *static* critical phenomena at the upper critical dimensionality in Euclidean space-time. This connection will pave the way to using the renormalization group to sum up infrared divergences non-perturbatively much in the same manner as in the theory of critical phenomena. In order to establish this connection more clearly we now introduce the dimensionless variable

$$\bar{P}^2 = \frac{P_E^2}{m_R^2} + 1.$$

The main reason for introducing this variable is that when $P_E^2 \rightarrow -m_R^2$, $\bar{P}^2 \rightarrow 0$; therefore the threshold behavior is mapped onto the zero Euclidean four momentum region in terms of the new variable. In critical phenomena logarithmic divergences appear when the Euclidean four momentum goes to zero at criticality at the upper critical dimension. We now introduce a wave function renormalization constant

$$Z_\phi(\bar{K}) = 1 + \lambda \ln \bar{K}^2$$

and a renormalized irreducible two-point function

$$\begin{aligned} \Gamma_R(\bar{P}, \bar{K}) &= Z_\phi(\bar{K}) \Gamma(\bar{P}) \\ &= m_R^2 \bar{P}^2 \left[1 + \lambda \left(\frac{1}{\bar{P}^2 - 1} \ln \bar{P}^2 + \ln \bar{K}^2 \right) \right] \end{aligned}$$

where λ is the dimensionless coupling (3.15) and \bar{K} is an arbitrary renormalization point. The bare irreducible function $\Gamma(\bar{P})$ is independent of the renormalization point \bar{K} , i.e. $\bar{K} d\Gamma/d\bar{K} = 0$, which leads to the renormalization group equation

$$\begin{aligned} \left[\bar{K} \frac{\partial}{\partial \bar{K}} - \eta \right] \Gamma_R(\bar{K}, \bar{P}) &= 0 \\ \eta &= \frac{\partial Z_\phi(\bar{K})}{\partial \ln \bar{K}} = 2\lambda. \end{aligned} \quad (3.17)$$

Near threshold, when $\bar{P} \rightarrow 0$,

$$\Gamma_R(\bar{K}, \bar{P}) = m_R^2 \bar{P}^2 \Phi\left(\frac{\bar{P}}{\bar{K}}\right)$$

with Φ a dimensionless function of its argument. The renormalization group equation (3.17) then leads to

$$\left[\bar{P} \frac{\partial}{\partial \bar{P}} + \eta \right] \Phi\left(\frac{\bar{P}}{\bar{K}}\right) = 0$$

with the solution

$$\Phi\left(\frac{\bar{P}}{\bar{K}}\right) = \Phi(1) \left[\frac{\bar{P}}{\bar{K}} \right]^{-\eta}$$

finally leading to the renormalization group improved two-point function

$$G_{RG}(k, s) = [\Phi(1) m_R^2 \bar{K}^2]^{-1} \left[\frac{\bar{K}^2}{\frac{P_E^2}{m_R^2} + 1} \right]^{1-\lambda} \quad (3.18)$$

which coincides with the one obtained by the Bloch-Nordsieck resummation, Eq. (3.14), up to an overall multiplicative factor. We can now retrace the same steps that led to the real-time evolution of the expectation value by performing the analytic continuation for the retarded correlation function and the inverse Laplace transform, leading to the relaxation with anomalous dimension given by Eq. (3.16).

The equivalence between the renormalization group improved and the Bloch-Nordsieck re-summed propagator is fairly well known [35,36]. The main purpose of our analysis is to make the point that the anomalous dimension in the amplitude of the expectation value can be understood as arising

from the scaling behavior of the Green's function near threshold in terms of the variable \bar{P} . This scaling behavior, a result of the infrared divergences associated with the emission of soft quanta, is akin to those in static critical phenomena.

Furthermore, this structure of anomalous dimension of the Euclidean propagator as a result of threshold infrared divergences is similar to that found in QED at $T=0$ either via a Bloch-Nordsieck or renormalization group resummation [35,36].

In preparation to the forthcoming discussion presented below on the real time interpretation of the renormalization group resummation of infrared divergences, it proves illuminating to obtain the perturbative form of the real time solution. This will allow us to identify the real time manifestation of infrared divergences. The naive perturbative expansion of the renormalization group improved propagator (3.14) to first order in the coupling λ near threshold leads obviously to the one-loop result (3.13). After performing the Fourier transform and obtaining the real time evolution given by Eq. (3.16) we can now expand naively in the coupling constant, and find

$$\phi_k(t) \approx A_k \cos(\omega_k t) [1 - \lambda \ln(\omega_k t)] + \dots \quad (3.19)$$

with A_k the amplitude read off from Eq. (3.16). This expression reveals that if we attempt a *perturbative* solution of the real-time equation of motion (3.1) we would find *logarithmic secular terms*, i.e. terms that grow in time and invalidate the perturbative solution at long times. In this case a perturbative expansion of the real-time equation of motion will break down at time scales $t_{break} \approx e^{1/\lambda}/\omega_k$. The renormalization group in the energy representation provides a resummation of the infrared divergences in the propagator, which leads to a real-time evolution that is asymptotically decreasing function of time.

We now study the perturbative solution of Eq. (3.1) that reveals indeed these secular terms, and implement a real-time version of the renormalization group that implements precisely this resummation.

B. Dynamical renormalization group at $T=0$

Having established the resummation of threshold infrared divergences both within the Bloch-Nordsieck approximation and the renormalization group, we now introduce a novel method that allows a similar resummation but *directly in real time*. Consider seeking a solution of the equation of motion (3.1) in perturbation theory in the coupling. Writing the self energy as an expansion in terms of the dimensionless coupling λ given by Eq. (3.15), $\Sigma_k = \sum_{n=1} \lambda^n \Sigma_k^{(n)}$ a perturbative solution obtained as a power series expansion is given by $\phi_k(t) = \phi_k^{(0)}(t) + \lambda \phi_k^{(1)}(t) + \dots$ with the hierarchy of equations

$$\ddot{\phi}_k^{(0)}(t) + \omega_k^2 \phi_k^{(0)}(t) = 0$$

$$\begin{aligned}\ddot{\phi}_k^{(1)}(t) + \omega_k^2 \phi_k^{(1)}(t) &= -[\delta m^2 + \gamma_k^{(1)}(0)]\phi_k^{(0)}(t) \\ &+ \int_0^t \gamma_k^{(1)}(t-t')\dot{\phi}_k^{(0)}(t')dt' \\ &\vdots = \vdots\end{aligned}$$

where we note that the contribution from the wave-function renormalization vanishes by virtue of the zeroth-order equation of motion. The solution to the zeroth-order equation is

$$\phi_k^{(0)}(t) = A_k e^{i\omega_k t} + A_k^* e^{-i\omega_k t}$$

and the initial conditions $\phi_k(t=0) = \phi_k(0)$, $\dot{\phi}_k(t=0) = 0$ implies $A_k = A_k^* = \phi_k(0)/2$, but we will leave both constants to recognize more easily the different contributions and we will use this condition at the end of the calculations. The solution to the above hierarchy of equations can be found in terms of the retarded Green's function of the unperturbed problem:

$$\mathcal{G}_R(t_1 - t_2) = \frac{1}{\omega_k} \sin[\omega_k(t_1 - t_2)] \Theta(t_1 - t_2).$$

The higher order corrections are easily (but tediously) computed using the spectral representation of the self-energy. The first order term is given by

$$\phi_k^{(1)}(t) = \phi_k^{(1,a)}(t) + \phi_k^{(1,b)}(t)$$

$$\begin{aligned}\phi_k^{(1,a)}(t) &= \frac{1}{\omega_k} \int \frac{d\omega}{\omega} \rho(\omega) \int_0^t dt' \int_0^{t'} dt_1 \sin[\omega_k(t-t')] \\ &\times \cos[\omega(t'-t_1)] \dot{\phi}^{(0)}(t_1)\end{aligned}$$

$$\begin{aligned}\phi_k^{(1,b)}(t) &= -\frac{1}{\omega_k} \left(\delta m^2 + \int \frac{d\omega}{\omega} \rho(\omega) \right) \\ &\times \int_0^t \sin[\omega_k(t-t')] \phi^{(0)}(t') dt'\end{aligned}$$

where the spectral density $\rho(\omega)$ is given to one loop order by Eq. (3.4) and its leading infrared contribution for $T \neq 0$ given by Eq. (3.6). The integral over the time variables can be done straightforwardly; the result is given by

$$\begin{aligned}\phi_k^{(1,a)}(t) &= -\frac{i}{4} \int \frac{d\omega}{\omega} \rho(\omega) \left\{ A_k e^{i\omega_k t} \left[\frac{1}{\omega_k - \omega} \left(t - \frac{e^{i(\omega - \omega_k)t} - 1}{i(\omega - \omega_k)} \right) + \omega \rightarrow -\omega \right] \right. \\ &- A_k e^{-i\omega_k t} \left[\frac{1}{\omega_k - \omega} \left(\frac{e^{2i\omega_k t} - 1}{2i\omega_k} - \frac{e^{i(\omega_k + \omega)t} - 1}{i(\omega_k + \omega)} \right) + \omega \rightarrow -\omega \right] \\ &+ A_k^* e^{i\omega_k t} \left[\frac{1}{\omega_k + \omega} \left(\frac{e^{-2i\omega_k t} - 1}{-2i\omega_k} + \frac{e^{-i(\omega_k - \omega)t} - 1}{i(\omega_k - \omega)} \right) + \omega \rightarrow -\omega \right] \\ &\left. - A_k^* e^{-i\omega_k t} \left[\frac{1}{\omega_k + \omega} \left(t - \frac{e^{i(\omega + \omega_k)t} - 1}{i(\omega + \omega_k)} \right) + \omega \rightarrow -\omega \right] \right\}\end{aligned}\quad (3.20)$$

$$\begin{aligned}\phi_k^{(1,b)}(t) &= \frac{i}{2\omega_k} \left(\delta m^2 + \int \frac{d\omega}{\omega} \rho(\omega) \right) \left\{ A_k e^{i\omega_k t} t - A_k^* e^{-i\omega_k t} t \right. \\ &- A_k e^{-i\omega_k t} \frac{e^{2i\omega_k t} - 1}{2i\omega_k} - A_k^* e^{i\omega_k t} \frac{e^{-2i\omega_k t} - 1}{2i\omega_k} \left. \right\}.\end{aligned}\quad (3.21)$$

Secular terms will arise from the contributions of the form $e^{i(\omega - \omega_k)t} - 1$ if the coefficients of these terms produce singularities in the integration region.

We are now in condition to analyze different cases. Although we are primarily interested in applying the method of dynamical renormalization group to the situation of infrared divergences, in order to gain insight and test this method we begin by studying situations in which results are known. To this purpose we address the familiar case of a generic interacting scalar theory in which the pole frequency ω_k is away

from thresholds (ω_{th}), either above, in which case there is a resonance, or below in which case the particle is stable.

1. $\omega_k \neq \omega_{th}$

For $\omega_k < \omega_{th}$ there are no singularities in the integration region; therefore the only secular terms are those linear in time in $\phi_k^{(1,a)}(t)$, $\phi_k^{(1,b)}(t)$. If on the other hand $\omega_k > \omega_{th}$ and far away from threshold, there are singularities (simple and double poles) in the integration region.

We can extract the secular terms in the above expression in both cases $\omega_k > \omega_{th}$ and $\omega_k < \omega_{th}$ by using the results of Appendix B or alternatively taking the long time limit using the distributions (see Appendix B)

$$\lim_{t \rightarrow \infty} \frac{1}{\alpha^2} [\alpha t - \sin \alpha t] = \mathcal{P} \left(\frac{1}{\alpha^2} \right) [\alpha t - \sin \alpha t]$$

$$\lim_{t \rightarrow \infty} \frac{1 - \cos \alpha t}{\alpha^2} = \pi t \delta(\alpha) + \mathcal{P}\left(\frac{1}{\alpha^2}\right)(1 - \cos \alpha t)$$

where the $\delta(\alpha)$ accounts for resonant denominators. This term is recognized from the familiar Fermi's golden rule.

Gathering the secular terms from both contributions (3.20),(3.21) and taking $A_k = A_k^*$ we find

$$\begin{aligned} \phi_k^{(1)}(t) = & A_k e^{i\omega_k t} \left\{ \left[i \frac{\delta m^2 + \Sigma_R(\omega_k)}{2\omega_k} t - \frac{\Sigma_I(\omega_k)}{2\omega_k} t \right] \right. \\ & + \int \frac{d\omega}{\omega} \rho(\omega) \mathcal{P} \frac{\omega_k^2}{(\omega_k^2 - \omega^2)^2} \\ & \left. - \int \frac{d\omega}{\omega} e^{i\omega t} \rho(\omega) \mathcal{P} \frac{\omega_k^2}{(\omega_k^2 - \omega^2)^2} \right\} + \text{c.c.} \end{aligned} \quad (3.22)$$

where we have used the expressions for the real and imaginary parts of the analytically continued self-energy given by Eqs. (3.9),(3.10). The last terms are non-secular at long times and remain perturbatively small.

In this manner the resummation of the secular terms is obvious and correspond to a shift in the pole position $\omega_k \rightarrow \omega_k + \delta\omega_k$ (finite by a proper choice of δm^2) and a width or decay rate Γ_k given by the imaginary and real part respectively of the first order correction above. Since this is the simplest and most familiar setting to introduce the dynamical renormalization group, we now present the resummation of the secular terms via this method. This is achieved by introducing a (complex) renormalization of the amplitude and writing [37,38]

$$A_k = \mathcal{A}_k(\tau) \mathcal{Z}_k(\tau) \quad (3.23)$$

$$\mathcal{Z}_k(\tau) = 1 + \lambda z_R^{(1)}(\tau) + i \lambda z_I^{(1)}(\tau) + \dots \quad (3.24)$$

where τ is an arbitrary time scale that acts as a renormalization point and the $z_{R,I}^{(n)}(\tau)$ are real functions. Choosing

$$\lambda z_R^{(1)}(\tau) = \frac{\Sigma_I(\omega_k)}{2\omega_k} \tau, \quad \lambda z_I^{(1)}(\tau) = -\frac{\delta m^2 + \Sigma_R(\omega_k)}{2\omega_k} \tau$$

we obtain

$$\begin{aligned} \phi_k(t) = & \mathcal{A}_k(\tau) e^{i\omega_k t} \left[1 + i \frac{\delta m^2 + \Sigma_R(\omega_k)}{2\omega_k} (t - \tau) \right. \\ & \left. - \frac{\Sigma_I(\omega_k)}{2\omega_k} (t - \tau) \right] + \text{c.c.} + \text{regular terms} \end{aligned} \quad (3.25)$$

where the regular terms refer to the non-secular last term. The meaning of the above expression is clear: a change in the time scale corresponds to a change in the (complex) amplitude of the expectation value. Whereas the original pertur-

bative expansion was only valid for times such that the contribution from the secular terms remain very small compared to the unperturbed value, the renormalized expression (3.25) remains valid for intervals $t - \tau$ such that the secular terms remain small. By choosing τ arbitrarily close to t we have improved the perturbative expansion. However, $\phi_k(t)$ does not depend on τ : a change of the renormalization point τ is compensated by a change in the complex amplitude \mathcal{A}_k . This leads to the *dynamical renormalization group equation* to lowest order:

$$\frac{\partial \mathcal{A}_k(\tau)}{\partial \tau} - \left[i \frac{\delta m^2 + \Sigma_R(\omega_k)}{2\omega_k} - \frac{\Sigma_I(\omega_k)}{2\omega_k} \right] \mathcal{A}_k(\tau) = 0$$

with the obvious solution

$$\mathcal{A}_k(\tau) = \mathcal{A}_k(0) e^{i\delta\omega_k \tau} e^{-\Gamma_k \tau}$$

$$\delta\omega_k = \frac{\delta m^2 + \Sigma_R(\omega_k)}{2\omega_k}$$

$$\Gamma_k = \frac{\Sigma_I(\omega_k)}{2\omega_k}.$$

Choosing the arbitrary scale τ to coincide with the time t we obtain the *resummed* expression for the expectation value:

$$\begin{aligned} \phi_k(t) = & \mathcal{C} \mathcal{A}_k(0) e^{i\omega_p(k)t} e^{-\Gamma_k t} - \mathcal{A}_k(0) \\ & \times \int \frac{d\omega}{\omega} e^{i\omega t} \rho(\omega) \mathcal{P} \frac{\omega_k^2}{(\omega_k^2 - \omega^2)^2} + \text{c.c.} \end{aligned} \quad (3.26)$$

$$\mathcal{C} = 1 + \int \frac{d\omega}{\omega} \rho(\omega) \mathcal{P} \frac{\omega_k^2}{(\omega_k^2 - \omega^2)^2}$$

with $\omega_p(k)$ the pole position shifted by one-loop corrections and Γ_k is identified with the decay or damping rate. The constant \mathcal{C} is the same as in Eq. (3.11). The on-shell renormalization leading to Eq. (3.11) is here a consequence of the perturbative expansion in terms of the solutions of the equations of motion.

After some straightforward algebra, the constant \mathcal{C} is found to be the same as the residue of the Laplace transform (3.7) at the pole (or resonance) at ω_p . The last, non-secular terms in Eq. (3.26), allows us to make contact with previous results [44]. The long time dynamics of this integral is dominated by the threshold contribution [44]. If the spectral density vanishes near threshold as $\rho(\omega) \approx (\omega - \omega_{th})^\alpha$, then the asymptotic time evolution is described by a power law relaxation $t^{-\alpha-1}$ (long time tails). Thus we see that the dynamical renormalization group resummation has obtained all of the features of the solution via the Laplace transform (3.7) which were previously obtained [44].

The resummation via the dynamical renormalization group has led to a (asymptotic) convergent perturbative expansion for the time evolution of the expectation value.

This simple case provides for a clear understanding of the resummations implied by the dynamical renormalization group and paves the way for understanding the more complicated cases of threshold singularities and finite temperature below.

2. Threshold singularities

Having established the reliability of the dynamical renormalization program in more familiar settings, we are now in position to apply this method to study the case of threshold infrared divergences arising from the emission of soft massless quanta. Thus we now return to the theory of a massive and a massless scalar fields of the beginning of this section. We begin by analyzing the situation at $T=0$ to make contact with the Bloch-Nordsieck and Euclidean renormalization group resummations, but now implementing the *dynamical renormalization group* resummation.

Since near threshold the spectral density (3.6) for $T=0$ becomes [λ is defined by Eq. (3.15)]

$$\rho(k, \omega) = -4\lambda\omega_k(\omega - \omega_k) + \mathcal{O}(\omega - \omega_k)^2,$$

besides the linear secular terms in $\phi_k^{(1,a)}, \phi_k^{(1,b)}$ [see Eqs. (3.20), (3.21)] there is a potentially infrared divergent term arising from $\phi_k^{(1,a)}$ which is *real* and using the results of Appendix B, found to be given by

$$\frac{1}{4} \int \frac{d\omega}{\omega} \rho(k; \omega) \frac{1 - \cos[(\omega - \omega_k)t]}{(\omega - \omega_k)^2}$$

$$\begin{aligned} & \mu t \gg 1 \\ & = -\lambda(\ln \mu t + \gamma) + \mathcal{O}(\lambda) \end{aligned}$$

where γ is Euler-Mascheroni constant (see details in Appendix B). We note that time acts as an infrared cutoff in the sense that for $\omega \approx \omega_k$ at finite time the integral is convergent. The infrared divergences are now manifest in a logarithmic time dependence. The linear secular terms combine just as in the previous case to provide an *imaginary* secular term given by $i[\delta m^2 + \Sigma_R(\omega_k)]/2\omega_k$ just as in Eq. (3.22) which in this case is simply frequency shift as can be seen from the expression for the self-energy given by Eq. (3.12). This shift is made finite with a proper choice of δm^2 . Thus in this case we find

$$\begin{aligned} \phi_k(t) &= A_k e^{i\omega_k t} \left[1 + i \frac{\delta m^2 + \Sigma_R(\omega_k)}{2\omega_k} t - \lambda \ln \bar{\mu} t \right] + \text{c.c.} \\ &+ \text{regular terms} \end{aligned}$$

with $\bar{\mu} = \mu e^\gamma$. Similarly to the previous case, we introduce the complex amplitude (3.24) and choose

$$\begin{aligned} \lambda z_R^{(1)}(\tau) &= \lambda \ln \bar{\mu} \tau; \\ \lambda z_I^{(1)}(\tau) &= -\frac{\delta m^2 + \Sigma_R(\omega_k)}{2\omega_k} \tau = -\delta\omega_k \tau \end{aligned}$$

leading to the following expression for $\phi_k(t)$:

$$\begin{aligned} \phi_k(t) &= A_k(\tau) e^{i\omega_k t} \left[1 + i\delta\omega_k(t - \tau) - \lambda \ln \frac{t}{\tau} \right] + \text{c.c.} \\ &+ \text{regular terms.} \end{aligned}$$

$\phi_k(t)$ is independent of the arbitrary time scale τ , leading to a renormalization group equation obeyed by the complex amplitude, which is now given this order by

$$\frac{\partial A_k(\tau)}{\partial \tau} - \left[i\delta\omega_k - \frac{\lambda}{\tau} \right] A_k(\tau) = 0$$

with the solution

$$A_k(\tau) = A_k(\tau_0) e^{i\delta\omega_k \tau} \left[\frac{\tau_0}{\tau} \right]^\lambda.$$

Again, choosing the scale τ to coincide with the time t , we finally obtain the asymptotic dynamics of the expectation value in this case to be given by

$$\phi_k(t) = A_k(\tau_0) e^{i\omega_k t} \left[\frac{\tau_0}{t} \right]^\lambda + \text{c.c.} + \text{small} \quad (3.27)$$

where $\omega_{k,R} = \omega_k + \delta\omega_k$. The terms denoted by small remain perturbative at all times and decay faster than the term with the anomalous dimension in weak coupling. This expression coincides with the long time behavior found in the previous sections via the Bloch-Nordsieck and the renormalization group resummation of the logarithmic infrared divergences of the propagator given by Eq. (3.16). We thus conclude that the dynamical renormalization group implements a resummation in *real time* which is complementary to the renormalization group or Bloch-Nordsieck resummations in the frequency representation of the propagator.

C. $T \neq 0$: Dynamical renormalization group resummation

At finite temperature the infrared divergences are enhanced by the Bose-Einstein distribution function of massless particles $N_q \approx T/q$, $q/T \ll 1$. This can be seen at the level of the spectral density $\rho(k; \omega)$ given by Eq. (3.6). Whereas the zero temperature contribution vanishes linearly near threshold, the finite temperature contribution remains constant there provided $m \neq 0$. In particular we find that near threshold the Laplace transform of the retarded propagator behaves as

$$\begin{aligned} \Gamma(k, s; T) &= [G(k, s; T)]^{-1} \\ &= (P_E^2 + m_R^2) \left[1 + \frac{g^2}{4\pi^2 P_E^2} \ln \left(\frac{P_E^2}{m_R^2} + 1 \right) \right] \\ &\quad - \bar{g}(T, k) \ln \frac{P_E^2 + m_R^2}{\mu^2} \end{aligned}$$

$$\bar{g}(T, k) = \frac{g^2}{4\pi^2} \left(\frac{T}{k} \right) \ln \frac{\omega_k + k}{\omega_k - k}$$

where μ is an arbitrary infrared cutoff scale and assumed that the external momenta is not too hard. Thus whereas the zero temperature inverse propagator actually vanishes at threshold (with an infrared divergent slope) the finite temperature propagator diverges there, reflecting the stronger infrared divergence at finite temperature. In this situation a Bloch-Nordsieck resummation of the Euclidean propagator is not clear as was emphasized in Ref. [32] and a multiplicative (wave function) renormalization cannot cure the infrared divergence since the finite temperature part is *not* proportional to $P_E^2 + m_R^2$.

It is precisely in this situation that the power of the dynamical renormalization group is revealed. We will study two different cases in detail: (i) hard external momentum $k \gg m$ (or $m=0$) and (ii) soft external momentum $k \leq m$.

1. Hard external momentum ($m \approx 0$)

In this case the Landau damping cut coalesces with the cut for $\omega \gg k$ and both terms of the spectral density (3.6) contribute. Because of the symmetry $\omega \rightarrow -\omega$ of the time dependent terms in Eq. (3.20) the frequency integral in the interval $-k < \omega < k$ plus the integral in the interval $k < \omega < \infty$ can be folded into an integral in the range $0 < \omega < \infty$ in terms of the effective finite temperature spectral density

$$\bar{\rho}(k; \omega) = -\frac{g^2}{\pi^2} \frac{T}{k} \ln \left[\frac{1 - e^{-|\omega+k|/2T}}{1 - e^{-|\omega-k|/2T}} \right] \Theta(\omega).$$

The asymptotic time dependence is dominated by the region $\omega \approx k$ which gives the infrared divergences of the propagator. In this region the effective spectral density

$$\bar{\rho}(k; \omega) \stackrel{\omega \rightarrow k}{=} \frac{g^2}{\pi^2} \frac{T}{k} \ln \frac{|\omega - k|}{2\bar{T}_k} \Theta(\omega) + \mathcal{O}(\omega - k),$$

$$\bar{T}_k = \frac{T}{1 - e^{-k/T}}. \quad (3.28)$$

We start by analyzing the different contributions to the infrared behavior of the coefficient of $A_k e^{i\omega_k t}$. A simple analysis leads to the following conclusions: (i) The contribution near threshold to the *imaginary* part of the coefficient cancels out between the production cut ($\omega > k$) and the Landau-damping cut ($0 < \omega < k$), leaving a linear secular term without infrared divergences that renormalizes the mass. (ii) The contributions near threshold to the *real* part are the same for both cuts $0 < \omega < k$ and $\omega > k$ and add up. This contribution in the asymptotic long time limit is obtained from the formulas in Appendix B and given by

$$\frac{g^2}{\pi^2} \frac{T}{2k^2} t \int_0^\infty \frac{dz}{z^2} \ln \left[\frac{z}{2t\bar{T}} \right] (1 - \cos z) = -\frac{g^2}{4\pi k^2} T t \ln \bar{\mu} t,$$

$$\bar{\mu} = 2\bar{T} e^{\gamma-1} \quad (3.29)$$

where we have quoted the leading contribution in the asymptotic time regime. Subleading terms can be consistently obtained using the formulas of Appendix B. The coefficient of $A_k^* e^{i\omega_k t}$ has a potential infrared divergence; however, the contribution, from both cuts cancel each other, leaving an infrared (and ultraviolet) finite result without secular terms. The zero temperature contribution and that of $\phi_k^{(1,b)}$ lead to a finite frequency shift. Implementing the dynamical renormalization group resummation we find

$$\phi_k(t) = \mathcal{A}_k(t_0) e^{i\omega_{k,R}(t-t_0)} e^{-a_k T t \ln[t/t_0]} + \text{c.c.}, \quad a_k = \frac{g^2}{4\pi k^2}$$

where we have solved the renormalization group equation with an initial condition at a time $t_0 = 1/\bar{\mu}$. This solution reveals clearly the renormalization group invariance, a change in the arbitrary time t_0 is compensated for by a change in the amplitude and an overall phase.

2. Soft external momentum ($m \neq 0$)

For $m \neq 0$ the Landau damping cut and the production cut are separated and the infrared divergences arise only from the production cut $\omega_k < \omega$. In the high temperature limit we find near threshold

$$\rho(k; \omega) \stackrel{\omega \rightarrow \omega_k}{=} -\frac{g^2}{\pi^2} \frac{T}{k} \ln \mathcal{M}_k [1 + \mathcal{O}(\omega - \omega_k)] \Theta(\omega - \omega_k) \quad (3.30)$$

$$\mathcal{M}_k \equiv 1 + \frac{2k}{m^2} (k + \omega_k).$$

We note that in this case the spectral density $\rho(k; \omega)$ approaches a constant value at threshold.

The terms proportional to $t/(\omega + \omega_k)$ in $\phi_k^{(1,a)}(t)$ do not have infrared divergences but they remain as secular terms that combine with those of $\phi_k^{(1,b)}(t)$ to give a renormalization of the frequency just as in the previous cases. The infrared divergences arise from terms with denominators $1/(\omega - \omega_k)$.

In this case these infrared divergences are manifest as logarithmic secular terms in the real and imaginary parts leading to damping and anomalous logarithmic phases.

These contributions are the following: (i) The *imaginary* part of the coefficient of $A e^{i\omega_k t}$ is given asymptotically by (see Appendix B)

$$\begin{aligned}
& \frac{it}{4} \int_{\omega_k}^{\infty} d\omega \frac{\rho(k; \omega)}{\omega} \frac{1}{\omega - \omega_k} \left[1 - \frac{\sin(\omega - \omega_k)t}{(\omega - \omega_k)t} \right] \\
&= - \frac{i g^2 \ln \mathcal{M}_k}{4 \pi^2 k \omega_k} T t \ln[\mu t e^{\gamma-1}] \\
&+ it \int_{\omega_k}^{\infty} \frac{d\omega}{\omega - \omega_k} \left[\frac{\rho(k; \omega)}{\omega} - \frac{\rho(k; \omega_k)}{\omega_k} \right. \\
&\quad \left. \times \Theta(\mu - \omega + \omega_k) \right] + \mathcal{O}\left(\frac{g^2}{t}\right). \quad (3.31)
\end{aligned}$$

Notice that Eq. (3.31) is *independent* of the scale μ as one can easily see since the derivative with respect to μ of the right-hand side (RHS) identically vanishes. The scale μ has been introduced just to have a dimensionless argument in the logarithms. (ii) The *real* part is asymptotically given by (see Appendix B for details)

$$\begin{aligned}
& \frac{1}{4} \int_{\omega_k}^{\infty} d\omega \frac{\rho(k; \omega)}{\omega} [1 - \cos(\omega - \omega_k)t] \left[\frac{1}{(\omega - \omega_k)^2} \right. \\
&\quad \left. - \frac{1}{2\omega_k(\omega - \omega_k)} \right] \\
&= - \frac{g^2 T t}{8 \pi k \omega_k} \ln \mathcal{M}_k - \frac{g^2 T}{4 \pi^2 m^2 \omega_k^2} \\
&\quad \times \left[1 + \frac{3 m^2}{2 k \omega_k} \ln \mathcal{M}_k \right] \ln[\bar{\mu} t e^{\gamma}] + \mathcal{O}(g^2).
\end{aligned}$$

The remaining secular but infrared safe terms in $\phi^{(1,a)}; \phi^{(1,b)}$ contribute to the imaginary part a term proportional to t which can be absorbed in a redefinition of the arbitrary scale μ in Eq. (3.31). The logarithmic divergence as $k/m \rightarrow \infty$ reflects precisely the logarithmic time dependence found in the previous case of hard momentum $m \approx 0$ and that results in the anomalous relaxation proportional to $t \ln \mu t$ as given by Eq. (3.29).

Implementing a resummation of the secular terms with the dynamical renormalization group following the steps outlined above leads to the asymptotic form

$$\phi_k(t) = \mathcal{A}_k(t_0) e^{i\varphi_k(t, t_0)} \left[\frac{t_0}{t} \right]^{\bar{b}_k} e^{-\bar{\Gamma}_k t} + \text{c.c.} \quad (3.32)$$

$$\varphi_k(t) = \omega_{k,R}(t - t_0) + \frac{g^2 \ln \mathcal{M}_k}{4 \pi^2 k \omega_k} T t \ln \frac{t}{t_0}$$

$$\bar{\Gamma}_k = \frac{g^2 T}{8 \pi k \omega_k} \ln \mathcal{M}_k$$

$$\bar{b}_k = \frac{g^2 T}{4 \pi^2 m^2 \omega_k} \left[1 + \frac{3 m^2}{2 k \omega_k} \ln \mathcal{M}_k \right].$$

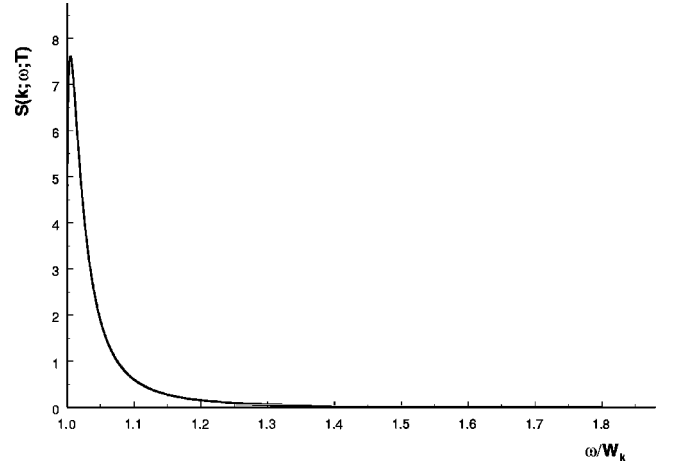


FIG. 1. The spectral density for the σ field $S(k, \omega; T)$ [Eq. (3.33)] near threshold for $\rho(k, \omega_k) = -0.005$.

It is interesting to try and understand the exponential damping in this case. For this we show in Fig. 1 the spectral density for the sigma field in this soft case near threshold:

$$\begin{aligned}
S(k, \omega; T) &= \frac{\Sigma_I(\omega, k; T)}{[\omega^2 - \omega_k^2 - \Sigma_R(\omega, k; T)]^2 + [\Sigma_I(\omega, k; T)]^2} \\
\Sigma_R(\omega, k; T) &= -\frac{1}{2} \rho(k, \omega_k) \ln \frac{\omega^2 - \omega_k^2}{\mu^2} + \mathcal{O}(\omega - \omega_k) \\
\Sigma_I(\omega, k; T) &= -\frac{\pi}{2} \rho(k, \omega_k) + \mathcal{O}(\omega - \omega_k) \quad (3.33)
\end{aligned}$$

where $\rho(k, \omega_k)$ is given by Eq. (3.30) and for the figure it was taken to be $\rho(k, \omega_k) = -0.005$. $S(k, \omega; T)$ vanishes at threshold but with a singular slope; this results in that the spectral density features a sharp peak near threshold, which is found to be at $\omega \approx \omega_k + \frac{1}{2} \rho(k, \omega_k) \ln |\rho(k, \omega_k)| + \dots$.

The decay rate is given by $(1/2) \Sigma_I(\omega_k)/(2\omega_k)$, the extra factor 1/2 being simply a result of the fact that ω_k is the threshold, i.e. the end point of the integral, and therefore the on-shell delta function only picks up half of the contribution. The logarithmic dependence of the phase is a consequence of the logarithmic infrared threshold divergences and prevents an interpretation in terms of quasiparticle poles.

This case must be contrasted to that of $m=0$, wherein the imaginary part of the self-energy is *infrared singular* at threshold. This is revealed in the logarithmic singularity in the limit $k/m \rightarrow \infty$ which reflects precisely the logarithmic time dependence found in the previous case of hard momentum $m \approx 0$, leading to the anomalous relaxation proportional to $t \ln[\mu t]$ as given by Eq. (3.29).

IV. DISCUSSION AND INTERPRETATION

Before proceeding further to the case of a gauge theory, it is convenient to pause and analyze the results that we have obtained so far and elucidate the main aspects of the dynamical renormalization group. The scalar model chosen in the

previous section is a non-trivial example of a superrenormalizable theory that displays the same type of infrared divergences as a critical theory at the upper critical dimension. The usual renormalization group leads to a Bloch-Nordsieck resummation and anomalous dimensions at $T=0$. Performing the Fourier transform we recognized that the real-time interpretation of the renormalization group resummation corresponds to a power law relaxation with the power determined by the anomalous dimension. A naive perturbative expansion in the dimensionless coupling results in secular terms, i.e. terms that grow in time and signal the breakdown of the perturbative expansion at long time scales. These same secular divergences are obtained directly in real time when the equation of motion for the expectation value is solved in a perturbative expansion. The dynamical renormalization group implements a resummation of these secular divergences which leads to an improved perturbative solution. The renormalization procedure can be understood with a very simple and pedagogical example, the weakly damped harmonic oscillator. Consider the equation of motion

$$\ddot{y} + y = -\epsilon \dot{y}, \quad \epsilon \ll 1.$$

Attempting to solve this equation in a perturbative expansion in ϵ leads to the lowest order solution (see Appendix C)

$$y(t) = A e^{it} \left[1 - \frac{\epsilon}{2} t \right] + \text{c.c.}$$

where the term that grows in time; i.e., the linear secular term leads to the breakdown of the perturbative expansion at time scales $t_{\text{break}} \propto 1/\epsilon$. The dynamical renormalization introduces a time scale τ in the form $A = A(\tau)Z(\tau)$, $Z(\tau) = 1 + \epsilon z_1(\tau) + \dots$; choosing z_1 to cancel the secular term at this time scale leads to the renormalization group equation

$$\frac{\partial A(\tau)}{\partial \tau} + \frac{\epsilon}{2} A(\tau) = 0$$

and the improved solution $y(t) = e^{-(\epsilon/2)t} [A(0)e^{it} + \text{c.c.}]$ after setting $\tau = t$ in the solution. This obviously is the correct solution to $\mathcal{O}(\epsilon)$. The interpretation of the renormalization group resummation is very clear in this simple example: the perturbative expansion is carried out to a time scale $\tau \ll 1/\epsilon$ within which perturbation theory is valid. The correction is recognized as a change in the amplitude, so at this time scale the correction is absorbed in a renormalization of the amplitude and the perturbative expansion is carried out to a longer time but in terms of the *amplitude at the renormalization scale*. The dynamical renormalization group equation is the differential form of this procedure of evolving in time, absorbing the corrections into the amplitude (and phases) and continuing the evolution in terms of the renormalized amplitudes and phases. This is the same spirit as the momentum-shell renormalization in critical phenomena. The details of the second order calculation and implementation of the renormalization group for this simple problem are offered in Appendix C to illustrate the shift in the frequency.

This interpretation in the simple exercise extends to the more complex situations with the same underlying mechanism; the secular divergences are absorbed in the complex amplitudes and the perturbative expansion is then carried in terms of the renormalized amplitudes. The differential form of this process is the dynamical renormalization group equation. Thus the similarities with the usual renormalization procedure are manifest.

In the same manner that the usual renormalization group sums the leading logarithms when the renormalization group functions are computed to one loop order, the dynamical renormalization group sums the leading secular terms when the coefficients are computed to lowest order. This is manifestly revealed by the naive perturbative expansion of the real time propagator (3.16).

The main objective of studying the different cases in the scalar theory of the previous section was to thoroughly *test* the method with a non-trivial example that can be studied with different methods. By studying these examples in detail we have learned that the dynamical renormalization group (i) provides a real-time equivalent of the resummation via the renormalization group in Euclidean space time (equivalent to Bloch-Nordsieck resummation) in the case of infrared threshold divergences at $T=0$ leading to relaxation with anomalous exponents, (ii) provides the usual mass shift and damping rates in the case of narrow resonances and (iii) leads to a similar resummation scheme at $T \neq 0$ when the threshold infrared divergences are more severe. This detailed analysis then provides confidence on this novel method to study the more interesting and relevant case of a gauge theory.

V. GAUGE THEORY: SQED

We are now in position to apply the method of the dynamical renormalization group to implement the resummation of infrared divergences in gauge theories, which is our primary goal.

We will study the case of scalar QED (SQED), since to lowest order in hard thermal loops this theory has the *same* properties as those of QED and QCD [10,18,29,32], in particular the infrared divergences associated with the propagation of the charged fields.

Since we are primarily interested in studying the real-time manifestation of the finite temperature infrared divergences, we will focus on the relaxation of the charged scalar field at finite temperature. Furthermore, we will only consider the contribution of transverse photons to the charged scalar self energy, since longitudinal photons are Debye screened at finite temperature ($m_D \approx eT$) and do not contribute to the infrared divergences.

In this Abelian theory it is rather straightforward to implement a gauge invariant formulation by projecting the Hilbert space on states annihilated by Gauss' law. Gauge invariant operators can be constructed and the Hamiltonian and Lagrangian can be written in terms of these. The resulting Lagrangian is exactly the same as that in Coulomb gauge [45] and is given by

$$\begin{aligned}\mathcal{L} = & \partial_\mu \Phi^\dagger \partial^\mu \Phi - m^2 \Phi^\dagger \Phi + \frac{1}{2} \partial_\mu \vec{A}_T \cdot \partial^\mu \vec{A}_T - e \vec{A}_T \cdot \vec{j}_T \\ & - e^2 \vec{A}_T \cdot \vec{A}_T \Phi^\dagger \Phi + \frac{1}{2} (\nabla A_0)^2 + e^2 A_0^2 \Phi^\dagger \Phi + e A_0 \rho, \\ \vec{j}_T = & i(\Phi^\dagger \vec{\nabla}_T \Phi - \vec{\nabla}_T \Phi^\dagger \Phi), \quad \rho = -i(\Phi \Phi^\dagger - \Phi^\dagger \Phi)\end{aligned}$$

where we have traded the instantaneous Coulomb interaction for a gauge invariant Lagrange multiplier field A_0 which should not be confused with a time component of the gauge field. \vec{A}_T is the transverse component satisfying $\vec{\nabla} \cdot \vec{A}_T(\vec{x}, t) = 0$. Since we are only interested in obtaining the infrared behavior arising from finite temperature effects, we do not introduce the renormalization counterterms to facilitate the study. The finite temperature behavior is ultraviolet finite. The non-equilibrium generating functional requires the fields on the forward and backward branches [45]. The equation of

motion for the charged scalar field is obtained by writing [43,45]

$$\Phi^\pm(\vec{x}, t) = \varphi(\vec{x}, t) + \Delta^\pm(\vec{x}, t), \quad \langle \Delta^\pm(\vec{x}, t) \rangle = 0$$

and similarly for the Hermitian conjugate fields.

In obtaining the equation of motion to one-loop order, we neglect the contribution from the Coulomb interaction. The reason as explained above is that the long-range Coulomb interaction will be screened by finite temperature effects with a Debye screening length $m_D \approx eT$ and hence the screened Coulomb interaction will be free of infrared divergences. However in an Abelian plasma the magnetic (transverse) photons are not screened (no static screening, only dynamical screening through Landau damping) and the exchange of soft magnetic photons will lead to threshold infrared divergences.

In terms of the spatial Fourier transform of $\varphi(\vec{x}, t)$ we find the equation of motion

$$\begin{aligned}\ddot{\varphi}(-\vec{k}, t) + (k^2 + m^2 + e^2 \langle \vec{A}_T^2(\vec{x}, t) \rangle) \varphi(-\vec{k}, t) - 4ie^2 \int_{-\infty}^{\infty} dt' \int \frac{d^3 q}{(2\pi)^3} k_T^i(\vec{q}) k_T^j(\vec{q}) \\ \times \{ [\langle A_T^{+i}(\vec{q}, t) A_T^{+j}(-\vec{q}, t') \rangle \langle \Delta^+(-\vec{k}-\vec{q}, t) \Delta^{\dagger,+}(\vec{k}+\vec{q}, t') \rangle \\ - \langle A_T^{+i}(\vec{q}, t) A_T^{-j}(-\vec{q}, t') \rangle \langle \Delta^+(-\vec{k}-\vec{q}, t) \Delta^{\dagger,-}(\vec{k}+\vec{q}, t') \rangle] \varphi(-\vec{k}, t') \} = J(-\vec{k}, t) \vec{k}_T(\vec{q}) = \vec{k} - \hat{q}(\vec{k} \cdot \hat{q})\end{aligned}$$

where we coupled an external source $J(\vec{k}, t) = J(\vec{k}) e^{\epsilon t} \Theta(-t)$ ($\epsilon \rightarrow 0^+$) to provide an initial value problem with an adiabatic switching on of the expectation value. The initial conditions for the expectation value are

$$\varphi(\vec{k}, t=0) = \varphi_k(0), \quad \dot{\varphi}(\vec{k}, t=0) = 0.$$

Infrared phenomena are associated with the soft limit of the intermediate photon, and therefore require the HTL resummation of the intermediate photon propagator. However, we begin our study of SQED by implementing the dynamical renormalization group to resum the perturbative expansion in the case in which the self-energy of the charged field only includes the exchange of a bare transverse photon. The study of this situation will shed light on the different physical processes that contribute and those that do not because of dynamical screening via Landau damping. In the next section we will include the HTL resummation of the exchanged photon and implement the dynamical renormalization group.

A. Bare photon propagators

The necessary free-field non-equilibrium Green's functions are given by scalar propagators

$$\langle \Phi^{(a)\dagger}(\vec{x}, t) \Phi^{(b)}(\vec{x}', t') \rangle = -i \int \frac{d^3 k}{(2\pi)^3} G_k^{ab}(t, t') e^{-i\vec{k} \cdot (\vec{x} - \vec{x}')} ,$$

where $(a, b) \in \{+, -\}$,

$$G_k^{++}(t, t') = G_k^>(t, t') \Theta(t - t') + G_k^<(t, t') \Theta(t' - t),$$

$$G_k^{--}(t, t') = G_k^>(t, t') \Theta(t' - t) + G_k^<(t, t') \Theta(t - t'),$$

$$G_k^{\pm\mp}(t, t') = -G_k^{<(>)}(t, t'),$$

$$G_k^>(t, t') = \frac{i}{2\omega_k} [(1 + n_k) e^{-i\omega_k(t-t')} + n_k e^{i\omega_k(t-t')}],$$

$$G_k^<(t, t') = \frac{i}{2\omega_k} [n_k e^{-i\omega_k(t-t')} + (1 + n_k) e^{i\omega_k(t-t')}],$$

$$\omega_k = \sqrt{\vec{k}^2 + m^2}, \quad n_k = \frac{1}{e^{\beta\omega_k} - 1}$$

photon propagators

$$\langle A_{Ti}^{(a)}(\vec{x}, t) A_{Tj}^{(b)}(\vec{x}', t') \rangle = -i \int \frac{d^3 k}{(2\pi)^3} \mathcal{G}_{ij}^{ab}(k; t, t') e^{-i\vec{k} \cdot (\vec{x} - \vec{x}')} ,$$

$$\begin{aligned}\mathcal{G}_{ij}^{++}(k; t, t') = & \mathcal{P}_{ij}(\vec{k}) [\mathcal{G}_k^>(t, t') \Theta(t - t') \\ & + \mathcal{G}_k^<(t, t') \Theta(t' - t)],\end{aligned}\quad (5.1)$$

$$\begin{aligned}\mathcal{G}_{ij}^{--}(k;t,t') &= \mathcal{P}_{ij}(\vec{k})[\mathcal{G}_k^{>}(t,t')\Theta(t'-t) \\ &\quad + \mathcal{G}_k^{<}(t,t')\Theta(t-t')], \\ \mathcal{G}_{ij}^{\pm\mp}(k;t,t') &= -\mathcal{P}_{ij}(\vec{k})\mathcal{G}_k^{<(>)}(t,t'),\end{aligned}\quad (5.2)$$

$$\begin{aligned}\mathcal{G}_k^{>}(t,t') &= \frac{i}{2k}[(1+N_k)e^{-ik(t-t')} \\ &\quad + N_k e^{ik(t-t')}],\end{aligned}$$

$$\begin{aligned}\mathcal{G}_k^{<}(t,t') &= \frac{i}{2k}[N_k e^{-ik(t-t')} \\ &\quad + (1+N_k)e^{ik(t-t')}],\end{aligned}$$

$$N_k = \frac{1}{e^{\beta k} - 1}.$$

Here $\mathcal{P}_{ij}(\vec{k})$ is the transverse projection operator:

$$\mathcal{P}_{ij}(\vec{k}) = \delta_{ij} - \frac{k_i k_j}{k^2}.$$

Finally we find the equation of motion for $t > 0$ to be given by

$$\begin{aligned}\ddot{\varphi}(-\vec{k},t) + [k^2 + M^2(T)]\varphi(-\vec{k},t) + \int_{-\infty}^t \Sigma(\vec{k},t-t')\varphi(-\vec{k},t')dt' &= 0 \\ \Sigma(\vec{k},t-t') &= -8e^2 k^2 \int \frac{d^3 q}{(2\pi)^3} \frac{1 - \cos^2 \theta}{4q\omega_{\vec{k}+\vec{q}}} \{ (1 + N_q + n_{\vec{k}+\vec{q}}) \sin(\omega_{\vec{k}+\vec{q}} + q)(t-t') + (N_q - n_{\vec{k}+\vec{q}}) \sin(\omega_{\vec{k}+\vec{q}} - q)(t-t') \} \\ M^2(T) &= m^2 + e^2 \langle \vec{A}^2 \rangle\end{aligned}$$

and θ is the angle between \vec{k} and \vec{q} . We see that the self-energy has a form very similar to that of the scalar case, Eq. (3.2), the only difference being the k^2 in front (reflecting the exchange of transverse photons) and the $1 - \cos^2 \theta$ inside the integral. Integrating by parts and using the initial conditions the resulting equation of motion can be written in the same form as in Eq. (3.1) with $\gamma_k(t)$ as in Eq. (3.5) in terms of the spectral density:

$$\begin{aligned}\rho(k;\omega) &= -\frac{e^2 k^2}{2\pi^2} \int \frac{q dq}{\omega_{k+q}} (1 - \cos^2 \theta) \\ &\quad \times d\cos \theta \{ (1 + N_q + n_{k+q}) \delta(\omega - q - \omega_{k+q}) \\ &\quad - (N_q - n_{k+q}) \delta(\omega + q + \omega_{k+q}) \}.\end{aligned}\quad (5.3)$$

As in the scalar case, we are mainly interested in the infrared effects associated with the emission and absorption of soft photons in the intermediate state and only the finite temperature contribution. Therefore we will (i) neglect the contribution of the distribution function of the intermediate charged scalar, i.e. the term n_{k+q} , and (ii) replace $N_q \approx T/q$ in the expression and neglect the vacuum contribution (the one), thus obtaining

$$\begin{aligned}\rho(k;\omega) &= -\frac{e^2 k T}{2\pi^2} \left\{ \int_{q^-}^{q^+} \frac{dq}{q} [1 - X^2(\omega, q)] \Theta(\omega - \omega_k) \right. \\ &\quad \left. - \int_{q^+}^{q^c} \frac{dq}{q} [1 - X^2(\omega, q)] \Theta(k^2 - \omega^2) \right\}\end{aligned}$$

with q^c an upper momentum cutoff $q^c \ll T$ and

$$\begin{aligned}X(\omega, q) &= \frac{\omega^2 - \omega_k^2 - 2\omega q}{2kq}, \\ q^\pm &= \frac{\omega^2 - \omega_k^2}{2(\omega \mp k)}.\end{aligned}$$

The second contribution with support below the light cone is identified with the Landau damping cut. Since the time dependent correlation functions involve the product of $\rho(k;\omega) \cos \omega t$, the frequency integral in the range $-k < \omega < 0$ arising from the Landau damping cut can be combined with the contribution from the positive frequency range by the symmetry of the integrand in Eq. (3.20). After a straightforward calculation we finally find the finite temperature contribution to the spectral density near threshold to be given by

$$\begin{aligned}\rho_{IR}(k;\omega) &= -\frac{e^2}{2\pi^2} \left(\frac{T}{k} \right) \left\{ (\omega^2 - k^2) \ln \left| \frac{\omega - k}{\omega + k} \right| + 2k\omega \right\} \\ &\quad \times [\Theta(\omega - \omega_k) + \Theta(k - \omega)] \Theta(\omega).\end{aligned}\quad (5.4)$$

There are several noteworthy features of these spectral density near threshold, as compared to the simpler case of the scalar theory studied in Sec. III [see Eq. (3.28)]. In this case the spectral density is constant at threshold. The term ω^2 [multiplying the logarithms in Eq. (5.4)] and the last term

$8\omega k$ arise from the region of the integral for which $\cos^2\theta \approx 1$ which would give a vanishing integrand were it not for the fact that there is a linear divergence as $\omega \rightarrow k$. These contributions arise from the emission and absorption of photons which are almost collinear with the incoming charged scalar.

We will now study the case of a massive scalar in two important limits: (i) $k/\omega_k = v \ll 1$ ($m \neq 0$), (ii) $k/\omega_k = v \approx 1$. In both cases the leading contribution is easily recognized to arise from the production cut $\omega \geq \omega_k$. For $m \neq 0$ there are no infrared divergences associated with the Landau damping cut.

I. $v \ll 1$

Since the spectral density is slowly varying near threshold, we can find the asymptotic behavior in time of the coefficient of $A_k e^{i\omega_k t}$ in Eq. (3.20) at large times following the same method as in previous sections and Appendix B. We find the following results: (i) The imaginary part of the coefficient of $A_k e^{i\omega_k t}$ is given by

$$\begin{aligned} & \frac{it}{4} \int_{\omega_k}^{\infty} d\omega \frac{\rho(k; \omega)}{\omega} \frac{1}{\omega - \omega_k} \left[1 - \frac{\sin(\omega - \omega_k)t}{(\omega - \omega_k)t} \right] \\ &= -\frac{i e^2 T}{8\pi^2} f(v) t \ln \bar{\mu} t + it \int_{\omega_k}^{\infty} \frac{d\omega}{\omega - \omega_k} \left[\frac{\rho(k; \omega)}{\omega} \right. \\ & \quad \left. - \frac{\rho(k; \omega_k)}{\omega_k} \Theta(\mu - \omega + \omega_k) \right] + \mathcal{O}\left(\frac{e^2 T}{t}\right) \end{aligned} \quad (5.5)$$

with

$$\begin{aligned} f(v) &= \frac{1-v^2}{v} \ln \frac{1-v}{1+v} + 2, \quad v = \frac{k}{\omega_k} \\ \bar{\mu} &= \mu e^{\gamma-1} \end{aligned} \quad (5.6)$$

the function $0 < f(v) < 2$ for $0 < v \leq 1$.

(ii) The real part of the coefficient of $A_k e^{i\omega_k t}$ in the asymptotically large time limit is given by (see Appendix B)

$$\begin{aligned} & \frac{1}{4} \int_{\omega_k}^{\infty} \frac{d\omega}{\omega} \rho(k; \omega) \frac{1 - \cos(\omega - \omega_k)t}{(\omega - \omega_k)^2} \\ &= -\Gamma_k t + \frac{2}{\pi \omega_k} \log(\mu t e^{\gamma}) \\ & \quad \times \left[\Gamma_k - \frac{1}{4} \frac{\partial \Sigma_I}{\partial \omega}(k, \omega_k) \right] + \mathcal{O}(e^2 T) \end{aligned} \quad (5.7)$$

with Γ_k given by

$$\Gamma_k = -\frac{\pi}{4} \frac{\rho(k; \omega_k)}{2\omega_k} = \frac{1}{2} \frac{\Sigma_I(\omega_k)}{2\omega_k} = \frac{\pi}{8} \left(\frac{e^2 T}{2\pi^2} \right) f(v) \quad (5.8)$$

where we used the expression (3.10) and the fact that for the production cut the spectral density only has support for positive frequencies. We then find the same phenomenon as in the previous case of the scalar particles in that the damping rate is one-half of the expected value. The reason again is that the full spectral density for the charged scalar is very similar to that featured in Fig. 1 with a prominent peak near threshold that is almost half of a Breit-Wigner peak. However, the logarithmic phase clearly exhibits the fact that cannot be interpreted as a quasiparticle resonance.

The contributions from the coefficient of $A_k^* e^{i\omega_k t}$ and the linear secular term from $\phi_k^{(1,b)}$ that contributes to the imaginary part are both subleading. We now renormalize the amplitude as in Eqs. (3.23), (3.24) with the choice

$$\begin{aligned} \lambda z_I^{(1)}(\tau) &= \frac{e^2 T}{8\pi^2} f(v) \tau \ln \bar{\mu} \tau \\ \lambda z_R^{(1)}(\tau) &= \Gamma_k \tau - \frac{2}{\pi \omega_k} \log(\mu \tau e^{\gamma}) \left[\Gamma_k - \frac{1}{4} \frac{\partial \Sigma_I}{\partial \omega}(k, \omega_k) \right]. \end{aligned}$$

The solution of the dynamical renormalization group equation now leads to the asymptotic behavior of the expectation value of the charged fields

$$\begin{aligned} \phi_k(t) &= \mathcal{A}_k(t_0) e^{i\varphi_k(t, t_0)} e^{-\Gamma_k(t-t_0)} \left(\frac{t_0}{t} \right)^{b_k} + \text{c.c.}, \\ \varphi_k(t, t_0) &= \omega_{k,R} t - \frac{e^2 T}{8\pi^2} f(v) t \ln \frac{t}{t_0}, \\ b_k &= \frac{2}{\pi \omega_k} \left[-\Gamma_k + \frac{1}{4} \frac{\partial \Sigma_I}{\partial \omega}(k, \omega_k) \right] \end{aligned} \quad (5.9)$$

where we integrated the dynamical renormalization group equation with initial condition at t_0 which is taken as some arbitrary renormalization point replacing the infrared cutoff $\bar{\mu}$. The renormalization group invariance of $\phi_k(t)$ is now explicit; a change of the arbitrary scale t_0 is compensated by a change in the amplitude $\mathcal{A}_k(t_0)$.

2. $v \rightarrow 1$

The limit $v \rightarrow 1$ must be studied carefully. As $\omega \rightarrow \omega_k \approx k$ the term $(\omega^2 - k^2) \ln|\omega - k|$ cannot be taken outside of the integral. However, upon the change of variable $\omega - k = z/t$ in the integral in the same manner as that leading to Eq. (5.7) leads to a term that is of the form $\ln[\mu t]/t$ for this contribution, which then becomes subleading compared with the term in $\rho_{I,R}(k; \omega)$ that does not vanish as $v \rightarrow 1$. The asymptotic large time behavior is therefore obtained from the previous section with $v \neq 1$ by simply setting $v = 1$. The contribution to the final result in this limit arises solely from the emission and absorption of collinear photons.

It is illuminating to try and understand this result in the hard limit $v \rightarrow 1$. The delta functions in Eq. (5.3) in the limit $k \gg m$ become $\delta(\omega - k - q \mp q \cos \theta)$; therefore as $\omega \rightarrow k$ the whole contribution arises from photons that are emitted or

absorbed collinearly $\theta=0,\pi$ with the moving (hard) scalar. However, as pointed out originally by Pisarski [20], the contribution from collinear photons does not survive screening effects arising from higher order contributions to the photon propagator. In particular, dynamical screening as a consequence of Landau damping of the intermediate photons cuts off the contribution of collinear photons, and leads to a greater contribution of photons emitted or absorbed at right angles with respect to the moving charged particle.

The analysis of this section is illustrative of the power of the dynamical renormalization group to obtain the asymptotic long time behavior. For the case of QED, QCD or SQED, the analysis presented in this section in terms of the bare propagators for the scalars and photons has very limited validity. For soft external momentum, the infrared region of the internal loop requires HTL resummation of the internal lines and vertices [20,32,29]. The main purpose of our analysis in this section, however, was to illustrate how the dynamical renormalization group is capable of revealing novel forms of relaxation with logarithmic corrections, power laws, anomalous dimensions, etc. These alternative forms of relaxation cannot be found by attempting to describe exponential relaxation and computing an imaginary part of the self-energy on shell.

We now include screening corrections via HTL resummation of internal lines. We will consider the case of hard external momentum for which only the internal photon line must be HTL resummed (the scalar is massive and hard).

B. Hard thermal loop-resummed photon propagators

We now focus on the case of hard external momentum of the charged scalar. In this case only the internal photon line receives HTL corrections, since the scalar in the loop is massive and hard; the vertex does not require resummation because one of the momenta into the vertex is hard (the scalar) [20,32,29]. Hence this situation is simpler than the case of soft external momentum that will be studied elsewhere.

In order to include the leading order screening effects in the photon propagator, we must use the hard thermal loop resummed propagators [20]. The generalization of the HTL resummation program in the Matsubara formulation of finite temperature field theory to the real time formulation is described in detail in Appendix A; we collect here only the main ingredients.

The photon propagators can be written as in Eqs. (5.1), (5.2) but now with the resummed Wightmann functions (see Appendix A)

$$\mathcal{G}_q^>(t-t') = \int dq_0 \tilde{\rho}_T(q_0, q) [1 + N(q_0)] e^{-iq_0(t-t')}$$

$$\mathcal{G}_q^<(t-t') = \int dq_0 \tilde{\rho}_T(q_0, q) N(q_0) e^{-iq_0(t-t')}$$

where in the hard thermal loop limit the spectral density for transverse photons is given by [29,45]

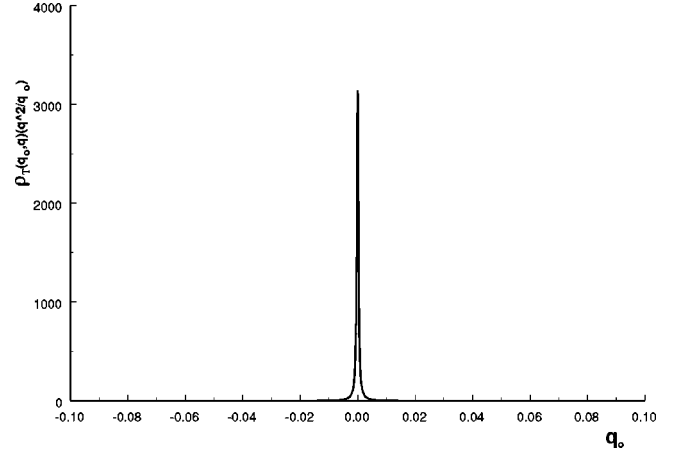


FIG. 2. $\rho_T(q_0, q)(q^2/q_0)$ for $q=0.1(eT/\sqrt{12})$ vs q_0 in units of $eT/\sqrt{12}$.

$$\tilde{\rho}_T(q_0, q) = \frac{1}{\pi} \frac{\Sigma_I(q_0, q) \Theta(q^2 - q_0^2)}{[q_0^2 - q^2 - \Sigma_R(q_0, q)]^2 + \Sigma_I^2(q_0, q)} + \text{sgn}(q_0) Z(q) \delta(q_0^2 - \omega_p^2(q))$$

$$\Sigma_I(q_0, q) = \frac{\pi e^2 T^2}{12} \frac{q_0}{q} \left(1 - \frac{q_0^2}{q^2} \right)$$

$$\Sigma_R(q_0, q) = \frac{e^2 T^2}{12} \left[2 \frac{q_0^2}{q^2} + \frac{q_0}{q} \left(1 - \frac{q_0^2}{q^2} \right) \ln \left| \frac{q_0 + q}{q_0 - q} \right| \right]$$

where $\omega_p(q)$ is the plasmon pole and $Z(q)$ its (momentum dependent) residue, which will not be relevant for the following discussion. Inserting these propagators in the expression for the self-energy, keeping only the term $N(q_0) \approx T/q_0$ in the resulting expressions, and focusing on the hard scalar limit $k \approx T \gg m$ we find that the self-energy can be written in the form of a dispersion relation just as in Eq. (3.3) but with

$$\rho(k; \omega) = -\frac{e^2 T k}{\pi^2} \int q^2 dq \int_{-1}^1 dX (1 - X^2) \times \int dq_0 \frac{\tilde{\rho}_T(q_0, q)}{q_0} \delta(\omega - k - q_0 - qX).$$

The infrared region corresponds to small q , and we find that for $q \ll eT$ the integrand $\tilde{\rho}_T(q_0, q)/q_0$ is strongly peaked at $q_0=0$. Figure 2 shows $\tilde{\rho}_T(q_0, q)q^2/q_0$ vs q_0 in units of $eT/\sqrt{12}$. We find that for $q \ll eT$ the photon spectral density is well approximated by [32]

$$\frac{\tilde{\rho}_T(q_0, q)}{q_0} = \frac{1}{\pi q^2} \frac{\Gamma}{\Gamma^2 + q_0^2}$$

$$\Gamma = q^3 \frac{12}{\pi e^2 T^2}.$$

When this Lorentzian distribution is integrated with smooth functions, it can be expanded in the width, obtaining

$$\frac{1}{\pi} \frac{\Gamma}{\Gamma^2 + q_0^2} \approx \delta(q_0) - \frac{\Gamma^2}{2} \delta''(q_0) + \dots$$

and the infrared behavior is dominated by the $\delta(q_0)$. Finally we find the infrared behavior of the spectral density of the self-energy to be given by

$$\rho(k; \omega) \approx -\frac{e^2 k T}{\pi^2} \int_{|\omega-k|}^{q^*} \frac{dq}{q} \left(1 - \frac{(\omega-k)^2}{q^2} \right) \approx \frac{e^2 k T}{\pi^2} \ln \frac{|\omega-k|}{\mu} \quad (5.10)$$

where $q^* \sim \mu \leq eT$ is an arbitrary upper momentum cutoff, which physically is of the order of the plasma frequency. We thus see that dynamical screening originating in Landau damping for the photon propagator suppresses the contribution of collinear photons and as $\omega \rightarrow k$ the contribution to the self-energy arises primarily from photons emitted or absorbed at right angles [20]. This is the same situation as in QED [32].

The final spectral density for the self-energy, Eq. (5.10), is therefore of the *same form* as the finite temperature self energy of the simple scalar theory studied in the earlier sections [see Eq. (3.28)] and thus justifies our excursion into that simpler theory.

We can now follow the same steps to study the secular terms in the real time perturbative expansion, which lead to Eq. (3.29).

The long time asymptotic behavior is obtained by inserting the spectral density, Eq. (5.10), in the expressions (3.20), (3.21) with $\omega_k = k$. The asymptotic dependence on time is extracted by changing variables to $\omega - k = z/t$ and upon setting $A_k = A_k^*$, we find that (i) the *imaginary* contribution to $\phi^{(1,a)} + \phi^{(1,b)}$ is given by an *infrared finite* and linear in time secular term which is interpreted as a renormalization of the frequency and (ii) the *real* contribution to $\phi^{(1,a)} + \phi^{(1,b)}$ is given by

$$\text{Re } \delta\phi_k(t) = -\alpha T t \ln \mu t, \quad \alpha = \frac{e^2}{4\pi}.$$

The first order correction is thus found to be given by

$$\phi_k^{(1)}(t) = A_k e^{i\omega_k t} [i\delta\omega_k t - \alpha T t \ln \mu t] + \text{c.c.} \\ + \text{regular perturbative terms.}$$

Introducing the renormalization of the amplitude as in Eqs. (3.23), (3.24) and choosing

$$\lambda z_R^{(1)}(\tau) = \alpha T \tau \ln \mu \tau, \quad \lambda z_R^{(2)}(\tau) = -\delta\omega_k \tau$$

we obtain the renormalization group equation

$$\frac{\partial \mathcal{A}_k(\tau)}{\partial \tau} - [i\delta\omega_k - \alpha T (\ln \mu \tau + 1)] \mathcal{A}_k(\tau) = 0$$

with the solution

$$\mathcal{A}_k(\tau) = \mathcal{A}_k(t_0) e^{i\delta\omega_k(\tau-t_0)} e^{-\alpha T \tau \ln(\tau/t_0)}$$

where we have chosen $t_0 = \mu^{-1}$ as an initial condition for the integration. Now choosing the arbitrary renormalization scale τ to coincide with the time t we finally arrive at one of the main results of this article which is the asymptotic behavior in time of the expectation value of the charged scalar field

$$\phi_k(t) = \mathcal{A}_k(t_0) e^{i\tilde{\omega}_k(t-t_0)} e^{-\alpha T t \ln[t/t_0]} + \text{c.c.} \quad (5.11)$$

thus displaying the renormalization group invariance of the solution: a change of the arbitrary time scale t_0 (inverse of the infrared cutoff) is compensated by a change in the amplitude. The quantity $\tilde{\omega}_k$ is the renormalized mass including the (infrared finite) HTL corrections.

A similar behavior for the asymptotic dynamics of the *fermion* field in QED has been obtained via the Bloch-Nordsieck resummation in Ref. [32]. The power of the dynamical renormalization group has now become explicit in that it transcends any approximation and implements a resummation of the logarithmic infrared divergences that in this case includes the resummation of the hard thermal loops.

VI. CONCLUSIONS, COMMENTS AND MORE QUESTIONS

In this article we have introduced a novel method of dynamical renormalization group resummation to study relaxation in real time. The first step of the program is to relate the retarded Green's functions that contain the dynamical information on the time evolution away from equilibrium in linear response to an initial value problem for the *expectation value* of the fields. This initial value problem in real time allows one to implement the method of dynamical renormalization, improving the perturbative solution by a resummation. This resummation is the real time counterpart of the resummation via the renormalization group in Euclidean field theory. We first apply our methods to a scalar theory of one massive and one massless field. The emission and absorption of the massless field introduced infrared divergences akin to those found in gauge theories. We compare in such a model the late behavior of the massive field amplitude at zero temperature [Eq. (3.27)] to results based on Bloch-Nordsieck resummations [Eq. (3.16)] as well as the renormalization group applied to Euclidean Green's functions [Eq. (3.18)].

Furthermore, for large temperature we compute the late behavior of the massive field amplitude both for hard and soft external momenta. For hard modes, the field amplitude relaxes as $e^{-(g^2 T/4\pi) t \ln[t/t_0]}$ where g stands for the coupling constant.

In real time the infrared divergences are manifest as secular terms in the perturbative solution of the evolution equation for the expectation value of the fields. The dynamical renormalization group implements a resummation of this secular terms that leads to an asymptotically convergent solution and clearly describes relaxation in real time.

After applying our method to the scalar model, we fo-

cused our attention on implementing the dynamical renormalization group to resum the infrared divergences associated with massless transverse photons in scalar QED at finite temperature. The infrared divergences in this theory are similar to those found in QED and in lowest order in QCD.

We have included the resummation of the hard thermal loops and Landau damping in the internal transverse photon propagators, and implemented a dynamical renormalization group resummation. The renormalization group improvement leads to an anomalous logarithmic relaxation for hard modes as a consequence of infrared divergences associated with the emission and absorption of photons at right angles. These anomalous logarithmic relaxations are similar to the scalar field behavior and consistent with those found in QED via the Bloch-Nordsieck resummation [32].

In all cases investigated (both in the scalar model and in QED) the field behavior prevents an interpretation of the relaxation of charged excitations in the medium in the form of a simple exponential with a damping rate determined by the imaginary part of the self-energy on shell.

The advantage of the dynamical renormalization group is that its implementation is rather simple and transcends any approximations of the Bloch-Nordsieck type; it can be consistently improved by considering higher orders in the hierarchy of equations obtained in perturbation theory.

Furthermore, the real time dynamics obtained via this resummation program leads to a clear interpretation of the relaxational processes and time scales *without* any assumptions about the validity of the quasiparticle picture of collective excitations. The analysis of secular terms in lowest order provides a simple criterion for deciding if the collective excitations can be described as narrow resonances with a width determined by the imaginary part of the self-energy on shell: linear secular terms lead to such a quasiparticle description; non-linear secular terms in lowest order signal anomalous, non-exponential relaxation.

The dynamical renormalization is a different resummation scheme than the HTL resummation, and the latter can be consistently included in the former as was shown in this article.

We are currently implementing this method to study relaxation of soft and hard fermion and gauge fields in QED and QCD directly in real time, thus bypassing the conceptual limitations of the quasiparticle picture in the sense of exponential relaxation and a damping rate determined by the imaginary part of the self-energy on shell. We expect to report results in the near future.

Comments and further questions. As we have seen in the example worked out in detail in Sec. III, at finite temperature the infrared divergences are akin to those of a *superrenormalizable* theory of critical phenomena, in the sense that they are no longer logarithmic because the temperature introduces a new scale. There are very few methods for renormalization of *infrared* divergences in superrenormalizable theories near the critical point, one of the most popular being the ϵ expansion where $\epsilon = d - 4$ is the departure from the upper critical dimension. Whereas the validity of the epsilon expansion appended by Padé resummation has been confirmed in Ising-like models via either strong coupling lattice expansion or

Monte Carlo simulations, the validity in a general case is at best questionable for $\epsilon = 1$. The ϵ expansion resums some subset of the Feynman diagrams and only some part of them [47] the leading logarithms.

The dynamical renormalization group therefore provides an alternative to study the infrared divergences directly in real time by resumming secular terms in the perturbative solution of the equation of motion. In the case of infrared divergences the secular terms reflect these in the form of logarithmic dependence on time. However, the usefulness of the dynamical renormalization group is not restricted to these logarithmic divergences, as explicitly shown in Sec. III; in the usual case of narrow resonances, the secular terms are linear (in lowest order) and their resummation through the renormalization group equation leads to the usual quasiparticle real-time evolution.

There is a translation between the resummation implied by the usual (Euclidean) renormalization group and that by the dynamical version: the Euclidean version sums the leading logarithms [36,47] and the dynamical version sums the leading secular terms [37].

It has been shown in Ref. [37] using the formal theory of envelopes that the dynamical renormalization group (RG) resummation of secular terms provides a uniform approximation to the exact solution for systems of ordinary differential equations. This is true to any given order of perturbation for arbitrary ordinary differential equations [37]. It will be very interesting to extend such a proof to the evolution equations considered in the present paper.

Thus, for the moment, the situation with the dynamical RG is similar to that of the ϵ expansion in critical phenomena for $\epsilon = 1$: it provides a resummation scheme for the infrared behavior in a consistent manner and it agrees with known results in cases where it can be compared. Furthermore, in the case in which the renormalization group and Bloch Nordsieck lead to non-trivial exponentiation of infrared divergences, the dynamical RG reproduces the results in real time. Thus we believe that the cases analyzed in detail in this article and those analyzed in the literature provide very strong evidence for the validity of this approach. The promise of the dynamical RG as a powerful method to study transport phenomena warrants a deeper study on the renormalization aspects of the evolution equations in real time and a more formal proof of the applicability of the dynamical RG in these problems. This avenue of study is currently in progress.

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APPENDIX A: EXACT RETARDED PROPAGATORS

In this appendix we gather and generalize some results of the HTL resummation program in Matsubara finite tempera-

ture field theory [10], to real time.

Consider the *exact* equilibrium Wightman and retarded Green's functions for a real scalar field:

$$\begin{aligned}
-iG_k^>(t-t') &= \langle \Phi_{\vec{k}}(t) \Phi_{-\vec{k}}(t') \rangle \\
&= -i \int d\omega \tilde{G}_k^>(\omega) e^{-i\omega(t-t')} \\
-iG_k^<(t-t') &= \langle \Phi_{-\vec{k}}(t') \Phi_{\vec{k}}(t) \rangle \\
&= -i \int d\omega \tilde{G}_k^<(\omega) e^{-i\omega(t-t')} \\
-iG_{R,k}(t-t') &= -i[G_k^>(t-t') - G_k^<(t-t')] \Theta(t-t') \\
&= -i \int \frac{dq_0}{2\pi} \tilde{G}_{R,k}(q_0) e^{-iq_0(t-t')}.
\end{aligned}$$

Inserting a complete set of eigenstates of the full interacting Hamiltonian, we obtain the spectral representations for the Fourier transforms given by

$$\begin{aligned}
-i\tilde{G}_k^>(\omega) &= \frac{1}{\mathcal{Z}} \sum_{m,n} e^{-\beta E_m} |\langle m | \Phi_k(0) | n \rangle|^2 \delta(\omega - (E_n - E_m)) \\
-i\tilde{G}_k^<(\omega) &= \frac{1}{\mathcal{Z}} \sum_{m,n} e^{-\beta E_n} |\langle m | \Phi_k(0) | n \rangle|^2 \delta(\omega - (E_m - E_n)) \\
&= e^{-\beta\omega} [-i\tilde{G}_k^>(\omega)]
\end{aligned} \tag{A1}$$

where the last equality, the Kubo-Martin-Schwinger (KMS) condition is obtained by relabelling $m \rightarrow n$ in the sum and $\mathcal{Z} = \sum_m e^{-\beta E_m}$ is the equilibrium partition function. Inserting a representation of the theta function we finally obtain

$$\begin{aligned}
\tilde{G}_{R,k}(q_0) &= - \int d\omega \frac{\tilde{\rho}(\omega, k)}{q_0 - \omega + i\epsilon} \\
\tilde{\rho}(\omega, k) &= -i[\tilde{G}_k^>(\omega) - \tilde{G}_k^<(\omega)] \\
&= \frac{1}{\pi} \text{Im}[\tilde{G}_{R,k}(\omega)].
\end{aligned}$$

Using the KMS condition (A1) we can finally write the *exact* non-equilibrium Wightmann functions in terms of the *exact* spectral density as follows:

$$\langle \Phi_{\vec{k}}(t) \Phi_{-\vec{k}}(t') \rangle = \int dq_0 \tilde{\rho}(q_0, k) [1 + N(q_0)] e^{-iq_0(t-t')}$$

$$\langle \Phi_{-\vec{k}}(t') \Phi_{\vec{k}}(t) \rangle = \int dq_0 \tilde{\rho}(q_0, k) N(q_0) e^{-iq_0(t-t')}$$

$$N(q_0) = \frac{1}{e^{\beta q_0} - 1}. \tag{A2}$$

Using the KMS condition, and relabelling the sum indices in the spectral representation we find that $\tilde{\rho}(q_0, k) = -\tilde{\rho}(-q_0, k)$. The same steps lead to an equivalent expression for the transverse gauge fields, whose Wightmann and Green's functions are proportional to the transverse projection operator.

The advantage of the representation (A2) is that once we compute the spectral function $\tilde{\rho}(q_0, k)$ in some approximation, we can insert the Wightmann functions in the internal loops thus providing a resummation of the perturbative series. The results from the first section allow to obtain the retarded propagator $\tilde{G}_{R,k}(q_0)$ from the solution of the initial value problem with an external source through the relation to linear response as detailed in the first section. For example by studying the equation of evolution for the expectation value of the transverse photon fields in the HTL approximation as was done in [45] we can obtain the spectral representation of the transverse fields in the HTL approximation and the results of this appendix allow us to implement a resummation of screened photon propagators into the real time description.

APPENDIX B: ASYMPTOTIC BEHAVIOR OF SPECTRAL INTEGRALS

We summarize in this appendix the late time behavior of integrals over the density of states used in Secs. III and IV.

In the formulas below $p(y)$ stands for a smooth function for $0 \leq y \leq \infty$. Here $p(0)$ as well as $p'(0)$ are finite. For large y , $p(y)$ decreases as a power such that the integrals over y converge at infinity. These properties are satisfied in all cases where these formulas were used in the paper:

$$\int_0^\infty \frac{dy}{y^2} (1 - \cos yt) p(y) \stackrel{t \rightarrow \infty}{=} \frac{\pi}{2} t p(0) + p'(0) [\ln(\mu t) + \gamma] + \int_0^\infty \frac{dy}{y^2} [p(y) - p(0) - y p'(0) \theta(\mu - y)] + \mathcal{O}\left(\frac{1}{t}\right),$$

$$\int_0^\infty \frac{dy}{y} (1 - \cos yt) p(y) \stackrel{t \rightarrow \infty}{=} p(0) [\ln(\mu t) + \gamma] + \int_0^\infty \frac{dy}{y} [p(y) - p(0) \theta(\mu - y)] + \mathcal{O}\left(\frac{1}{t}\right),$$

$$\int_0^\infty \frac{dy}{y} \left(t - \frac{\sin yt}{y} \right) p(y) \stackrel{t \rightarrow \infty}{=} t p(0) [\ln(\mu t) + \gamma - 1] + t \int_0^\infty \frac{dy}{y} [p(y) - p(0) \theta(\mu - y)] + \mathcal{O}\left(\frac{1}{t}\right),$$

where $\gamma=0.5772157\dots$ is Euler's constant.

Notice that the formulas are *independent* of the scale μ as one can easily see since the derivative with respect to μ of the RHS identically vanishes. The scale μ has been introduced just to have a dimensionless argument in the logarithms.

We have also used similar integrals when the resonance was away from the threshold. We have, for such a case,

$$\int_{-A}^{\infty} \frac{dy}{y^2} (1 - \cos yt) p(y) \stackrel{t \rightarrow \infty}{=} \pi t p(0) + \mathcal{P} \int_{-A}^{\infty} \frac{dy}{y^2} [p(y) - p(0)] + \mathcal{O}\left(\frac{1}{t}\right),$$

$$\int_{-A}^{\infty} \frac{dy}{y} \left(t - \frac{\sin yt}{y}\right) p(y) \stackrel{t \rightarrow \infty}{=} t \mathcal{P} \int_{-A}^{\infty} \frac{dy}{y} p(y) - \pi p'(0) + \mathcal{O}\left(\frac{1}{t}\right)$$

where A is a fixed positive number.

In addition, we needed in Secs. III and IV integrals where the spectral density has a logarithmic singularity at a finite point:

$$\int_{-A}^{\infty} \frac{dy}{y^2} (1 - \cos yt) p(y) \ln \frac{|y|}{2\bar{T}} \stackrel{|y| \rightarrow \infty}{=} \pi t p(0) [1 - \gamma - \ln(2 t \bar{T})] + \mathcal{P} \int_{-A}^{\infty} \frac{dy}{y^2} [p(y) - p(0)] \ln \frac{|y|}{2\bar{T}} + \mathcal{O}\left(\frac{1}{t}\right),$$

$$\int_{-A}^{\infty} \frac{dy}{y} \left(t - \frac{\sin yt}{y}\right) p(y) \ln \frac{|y|}{2\bar{T}} \stackrel{|y| \rightarrow \infty}{=} t \mathcal{P} \int_{-A}^{\infty} \frac{dy}{y} p(y) \ln \frac{|y|}{2\bar{T}} - \pi p'(0) [\ln(2 t \bar{T}) + \gamma] + \mathcal{O}\left(\frac{1}{t}\right).$$

APPENDIX C: A SIMPLE EXAMPLE: THE DAMPED HARMONIC OSCILLATOR

In this appendix we give a rather simple example of the dynamical renormalization group for pedagogical reasons and to illustrate the fundamental features within a simple setting. We consider the equation of motion of a damped harmonic oscillator,

$$\ddot{y} + y = -\epsilon \dot{y}, \quad \epsilon \ll 1 \quad (\text{C1})$$

and seek a solution in a perturbative expansion in ϵ of the form $y = y_0 + \epsilon y_1 + \epsilon^2 y_2 + \dots$ where the y_i are solutions to the following hierarchy of equations:

$$\begin{aligned} \ddot{y}_0 + y_0 &= 0, \\ \ddot{y}_1 + y_1 &= -\dot{y}_0, \\ \ddot{y}_2 + y_2 &= -\dot{y}_1, \\ &\vdots \end{aligned}$$

These equations can be solved iteratively by starting from the zero order solution

$$y_0(t) = A e^{it} + \text{c.c.},$$

in terms of the retarded Green's function

$$G_{\text{ret}}(t-t') = \sin(t-t') \theta(t-t').$$

Up to second order in ϵ , the solution is given by

$$y(t) = A e^{it} \left[1 - \frac{\epsilon}{2} t + \frac{\epsilon^2}{8} t^2 + i \frac{\epsilon^2}{8} t \right] + \text{c.c.} + \text{non-secular}.$$

Note that this solution contains secular terms that grow in t ; the terms denoted by *non-secular* remain finite at all times. We see that the perturbative expansion breaks down at a time scale $\approx 1/\epsilon$. The expression in the brackets can be interpreted as a change in the complex amplitude. The dynamical renormalization is achieved by introducing a time scale τ ; at which the secular terms are absorbed in a renormalization of the complex amplitude. We write $A = A(\tau) Z(\tau)$ with $Z(\tau) = 1 + \epsilon z_1(\tau) + \epsilon^2 z_2(\tau)$ and choose $z_i(\tau)$ to cancel the secular terms at the scale τ , this is similar to choosing the renormalization scale in the usual renormalization program. Up to $\mathcal{O}(\epsilon^2)$ we find

$$z_1(\tau) = \frac{\tau}{2}, \quad z_2(\tau) = \frac{\tau^2}{8} - i \frac{\tau}{8}.$$

After renormalization the solution is given by

$$y(t, \tau) = A(\tau) e^{it} \left[1 - \frac{\epsilon}{2} (t - \tau) + \frac{\epsilon^2}{8} (t - \tau)^2 + i \frac{\epsilon^2}{8} (t - \tau) \right] + \text{c.c.} + \text{non-secular}. \quad (\text{C2})$$

Since τ is an arbitrary scale, the solution cannot depend on it; thus the statement $dy(t, \tau)/d\tau = 0$ leads to the dynamical renormalization group equation to this order:

$$\frac{\partial A(\tau)}{\partial \tau} + A(\tau) \left(\frac{\epsilon}{2} - i \frac{\epsilon^2}{8} \right) = 0$$

where we have expanded the τ derivative of the amplitude in a power series expansion in ϵ consistently to second order. Obviously the solution to the renormalization group equation is given by

$$A(\tau) = A(0)e^{-(\epsilon/2)\tau}e^{i(\epsilon^2/8)\tau}.$$

Setting $t = \tau$ in Eq. (C2) we finally find

$$y(t) = A(0)e^{-(\epsilon/2)t}e^{i(1-\epsilon^2/8)t} + \text{c.c.}$$

which is obviously the correct solution to second order. Further simple and not-so-simple examples can be found in Ref. [37].

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